Introduction to Representations of the Canonical Commutation and Anticommutation Relations

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1 Introduction

Since the early days of quantum mechanics it has been noted that the position operator x and the momentum operator $D := -\mathrm{i}\nabla$ satisfy the following commutation relation:

$$[x, D] = \mathrm{i}. \tag{1}$$

Similar commutation relation hold in the context of the second quantization. The bosonic creation operator a^* and the annihilation operator a satisfy

$$[a, a^*] = 1. \tag{2}$$

If we set $a^* = \frac{1}{\sqrt{2}}(x - \mathrm{i}D)$, $a = \frac{1}{\sqrt{2}}(x + \mathrm{i}D)$, then (1) implies (2), so we see that both kinds of commutation relations are closely related.

Strictly speaking the formulas (1) and (2) are ill defined because it is not clear how to interpret the commutator of unbounded operators. Weyl proposed to replace (1) by

$$\mathrm{e}^{\mathrm{i}\eta x}\mathrm{e}^{\mathrm{i}qD} = \mathrm{e}^{-\mathrm{i}q\eta}\mathrm{e}^{\mathrm{i}qD}\mathrm{e}^{\mathrm{i}\eta x}, \ \eta, q \in \mathbb{R}, \tag{3}$$

which has a clear mathematical meaning [75]. (1) is often called the canonical commutation relation (CCR) in the Heisenberg form and (3) in the Weyl form.

It is natural to ask whether the commutation relations fix the operators x and D uniquely up to the unitary equivalence. If we assume that we are given two self-adjoint operators x and D acting irreducibly on a Hilbert space and satisfying (3), then the answer is positive, as proven by Stone and von Neumann [51], see also [68].

It is useful to generalize the relations (1) and (2) to systems with many degrees of freedom. They were used by Dirac to describe quantized electromagnetic field in [30].

In systems with many degrees of freedom it is often useful to use a more abstract setting for the CCR. One can consider n self-adjoint operators $\phi_1, \dots, \phi_n$ satisfying relations

$$[\phi_j, \phi_k] = \mathrm{i}\omega_{jk}, \tag{4}$$

where ω_{jk} is an antisymmetric matrix. Alternatively one can consider the Weyl (exponentiated) form of (4) satisfied by the so-called Weyl operators $\mathrm{e}^{\mathrm{i}(y_1\phi_1+\cdots+y_n\phi_n)}$, where $(y_1, \dots, y_n) \in \mathbb{R}^n$.

The Stone-von Neumann Theorem about the uniqueness can be extended to the case of regular representations of the CCR in the Weyl form if ω_{jk} is a finite dimensional symplectic matrix. Note that in this case the relations (4) are invariant with respect to the symplectic group. This invariance is implemented by a projective unitary representation of the symplectic group, as has been noted by Segal [63]. It can be expressed in terms of a representation of the two-fold covering of the symplectic group — the so-called metaplectic representation, [74, 61].

The symplectic invariance of the CCR plays an important role in many problems of quantum theory and of partial differential equations. An interesting and historically perhaps first nontrivial application is due to Bogolubov, who used it in the theory of superfluidity of the Bose gas [15]. Since then, the idea of using symplectic transformations in bosonic systems often goes in the physics literature under the name of the Bogolubov method, see e.g. [34].

The Canonical Anticommutation Relations (CAR) appeared in mathematics before quantum theory in the context of Clifford algebras [19]. A Clifford algebra is the associative algebra generated by elements $\phi_1, \dots, \phi_n$ satisfying the relations

$$[\phi_i, \phi_j]_+ = 2\delta_{ij}, \tag{5}$$

where $[\cdot, \cdot]_+$ denotes the anticommutator. It is natural to assume that the ϕ_i are self-adjoint. It is not difficult to show that if the representation (5) is irreducible, then it is unique up to the unitary equivalence for n even and that there are two inequivalent representations for an odd n.

In quantum physics, the CAR appeared in the description of fermions [45]. If $a_1^*, \dots, a_m^*$ are fermionic creation and $a_1, \dots, a_m$ fermionic annihilation operators, then they satisfy

$$[a_i^*, a_j^*]_+ = 0, \quad [a_i, a_j]_+ = 0, \quad [a_i^*, a_j]_+ = \delta_{ij}.$$

If we set $\phi_{2j-1} := a_j^* + a_j$, $\phi_{2j} := -\mathrm{i}(a_j^* - a_j)$, then we see that they satisfy the relations (5) with $n = 2m$.

Another application of the CAR in quantum physics are Pauli [52] and Dirac [31] matrices used in the description of spin $\frac{1}{2}$ particles.

Clearly, the relations (5) are preserved by orthogonal transformations applied to $(\phi_1, \dots, \phi_n)$. The orthogonal invariance of the CAR is implemented by a projective unitary representation. It can be also expressed in terms of a

representation of the double covering of the orthogonal group, called the Pin group. The so-called spinor representations of orthogonal groups were studied by Cartan [18], and Brauer and Weyl [12]

The orthogonal invariance of the CAR relations appears in many disguises in algebra, differential geometry and quantum physics. In quantum physics it is again often called the method of Bogolubov transformations. A particularly interesting application of this method can be found in the theory of superfluidity (a version of the BCS theory that can be found e.g. in [34]).

The notion of a representation of the CCR and CAR gives a convenient framework to describe Bogolubov transformations and their unitary implementations. Analysis of Bogolubov transformations becomes especially interesting in the case of an infinite number of degrees of freedom. In this case there exist many inequivalent representations of the CCR and CAR, as noticed in the 50's, e.g. by Segal [64] and Gaarding and Wightman [38].

The most commonly used representations of the CCR/CAR are the so-called Fock representations, defined in bosonic/fermionic Fock spaces. These spaces have a distinguished vector Ω called the vacuum killed by the annihilation operators and cyclic with respect to creation operators. They were introduced in quantum physics by Fock [35] to describe systems of many particle systems with the Bose/Fermi statistics. Their mathematical structure, and also the essential self-adjointness of bosonic field operators, was established by Cook [21].

The passage from a one particle system to a system with an arbitrary number of particles subject to the Bose/Fermi statistics is usually called second quantization. Early mathematical research on abstract aspects of second quantization was done by Friedrichs [37] and Segal [63, 64].

In the case of an infinite number of degrees of freedom, the symplectic/orthogonal invariance of representations of the CCR/CAR becomes much more subtle. The unitary implementability of symplectic/orthogonal transformations in the Fock space is described by the Shale/Shale-Stinespring Theorem. These theorems say that implementable symplectic/orthogonal transformation belong to a relatively small group $Sp_2(\mathcal{Y})/O_2(\mathcal{Y})$, [61]/[62]. In the case of an infinite number of degrees of freedom there also exists an analogue of the metaplectic/Pin representation. This seems to have been first noted by Lundberg [49].

Among early works describing these results let us mention the book by Berezin [14]. It gives concrete formulas for the implementation of Bogolubov transformations in bosonic and fermionic Fock spaces. Related problems were discussed, often independently, by other researchers, such as Ruijsenaars [59, 60].

As stressed by Segal [64], it is natural to apply the language of C^*-algebras in the description of the CCR and CAR. This is easily done in the case of the CAR, where there exists an obvious candidate for the C^*-algebra of the CAR over a given Euclidean space [17]. If this Euclidean space is of countably infinite dimension, the C^*-algebra of the CAR is isomorphic to the so called

$UHF(2^\infty)$ algebra studied by Glimm. Using representations of this C^*-algebra one can construct various non-isomorphic kinds of factors (W^*-algebras with a trivial center), studied in particular by Powers [54] and Powers and Stœrmer [55].

In the case of the CCR, the choice of the corresponding C^*-algebra is less obvious. The most popular choice is the C^*-algebra generated by the Weyl operators, studied in particular by Slawny [66]. One can, however, argue that the "Weyl algebra" is not very physical and that there are other more natural choices of the C^*-algebra of the CCR. Partly to avoid discussing such (quite academic) topics, in our lecture notes we avoid the language of C^*-algebras. On the other hand, we will use the language of W^*-algebras, which seems natural in this context.

One class of representations of the CCR and CAR – the quasi-free representations – is especially widely used in quantum physics. In mathematical literature they have been first identified by Robinson [58] and Shale and Stinespring [62]. Quasi-free representations were extensively studied, especially by Araki [2, 4, 5, 7, 10] and van Daele [22].

A concrete realization of quasi-free representations is furnished by the so-called Araki-Woods representations [8] in the bosonic and Araki-Wyss representations [9] in the fermionic case. We describe these representations in detail. From the physical point of view, they can be viewed as a kind of a thermodynamical limit of representations for a finite number of degrees of freedom. From the mathematical point of view, they provide interesting and physically well motivated examples of factors of type II and III. It is very instructive to use the Araki-Woods and Araki-Wyss representations as illustrations for the Tomita-Takesaki theory and for the so-called standard form of a W^*-algebra [42] (see also [4, 20, 16, 67, 29]). They are quite often used in recent works on quantum statistical physics, see e.g. [27, 44]

It is interesting to note that the famous paper of Haag, Hugenholtz and Winnink [41] about the KMS condition was directly inspired by the Araki-Woods representation.

Araki-Woods/Araki-Wyss representations can be considered also in the case of a finite number of degrees of freedom. In this case, they are equivalent to a multiple of the usual representations of the CCR/CAR. This equivalence can be described by the GNS representation with respect to a quasi-free state composed with an appropriate unitarily implemented Bogolubov transformation. We discuss this topic in the section devoted to "confined" Bose/Fermi gas.

It is easy to see that real subspaces of a complex Hilbert space form a complete complemented lattice, where the complementation is given by the symplectic orthogonal complement. It is also clear that von Neumann algebras on a given Hilbert space form a complete complemented lattice with the commutant as the complementation. It was proven by Araki [1] (see also [32]) that von Neumann algebras on a bosonic Fock space associated to real subspaces of the classical phase space also form a complemented complete

lattice isomorphic to the corresponding lattice of real subspaces. We present this result, used often in algebraic quantum field theory. We also describe the fermionic analog of this result (which seems to have been overlooked in the literature).

In the last section we describe a certain class of operators that we call Pauli-Fierz operators, which are used to describe a small quantum system interacting with a bosonic reservoir, see [25, 26, 27, 11] and references therein. These operators have interesting mathematical and physical properties, which have been studied in recent years by various authors. Pauli-Fierz operators provide a good opportunity to illustrate the use of various representations of the CCR.

The concepts discussed in these lectures, in particular representations of the CCR and CAR, constitute, in one form or another, a part of the standard language of mathematical physics. More or less explicitly they are used in any textbook on quantum field theory. Usually the authors first discuss quantum fields "classically"—just the relations they satisfy without specifying their representation. Only then one introduces their representation in a Hilbert space. In the zero temperature, it is usually the Fock representation determined by the requirement that the Hamiltonian should be bounded from below,see e.g. [24]. In positive temperatures one usually chooses the GNS representation given by an appropriate thermal state.

The literature devoted to topics contained in our lecture notes is quite large. Let us mention some of the monographs. The exposition of the C^*-algebraic approach to the CCR and CAR can be found in [17]. This monograph provides also extensive historical remarks. One could also consult an older monograph [33]. Modern exposition of the mathematical formalism of second quantization can be also found e.g. in [39, 13]. We would also like to mention the book by Neretin [50], which describes infinite dimensional metaplectic and Pin groups, and review articles by Varilly and Gracia-Bondia [72, 73]. A very comprehensive article devoted to the CAR C^*-algebras was written by Araki [6]. Introductions to Clifford algebras can be found in [48, 71, 47]. In this collection of lecture notes De Bièvre discusses the localizability for bosonic fields [24].

The theory of the CCR and CAR involves a large number of concepts coming from algebra, analysis and physics. This is why the literature about this subject is very scattered and uses various conventions, notations and terminology. Even the meaning of the expressions "a representation of the CCR" and "a representation of the CAR" depends slightly on the author.

In our lectures we want to stress close analogies between the CCR and CAR. Therefore, we tried to present both formalisms in a possibly parallel way.

We also want to draw the reader's attention to W^*-algebraic aspects of the theory. They shed a lot of light onto some aspects of mathematical physics. The CAR and CCR are also a rich source of illustrations for various concepts of the theory of W^*-algebras.

We often refrain from giving proofs. Most of them are quite elementary and can be easily provided by the interested reader.

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2 Preliminaries

In this section we review our basic notation, mostly about vector spaces and linear operators.

2.1 Bilinear forms

Let α be a bilinear form on a vector space $\mathcal{Y}$. The action of α on a pair of vectors $y_1, y_2 \in \mathcal{Y}$ will be written as $y_1 \alpha y_2$. We say that a linear map r on $\mathcal{Y}$ preserves α iff

$$(ry_1)\alpha(ry_2) = y_1 \alpha y_2, \quad y_1, y_2 \in \mathcal{Y}.$$

We say that α is nondegenerate, if for any non-zero $y_1 \in \mathcal{Y}$ there exists $y_2 \in \mathcal{Y}$ such that $y_2 \alpha y_1 \neq 0$.

An antisymmetric nondegenerate form is called symplectic. A symmetric nondegenerate form is called a scalar product.

2.2 Operators in Hilbert spaces

The scalar product of two vectors Φ, Ψ in a Hilbert space will be denoted by $(\Phi|\Psi)$. It will be antilinear in the first argument and linear in the second.

If $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, then $B(\mathcal{H}_1, \mathcal{H}_2)$, resp. $U(\mathcal{H}_1, \mathcal{H}_2)$ denotes bounded, resp. unitary operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

A^* denotes the hermitian adjoint of the operator A.

An operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ is called antiunitary iff it is antilinear, bijective and $(U\Phi|U\Psi) = \overline{(\Phi|\Psi)}$.

$B^2(\mathcal{H}_1, \mathcal{H}_2)$ denotes Hilbert-Schmidt operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, that is $A \in B^2(\mathcal{H}_1, \mathcal{H}_2)$ iff $\mathrm{Tr} A^* A < \infty$. Note that $B^2(\mathcal{H}_1, \mathcal{H}_2)$ has a natural structure of the Hilbert space with the scalar product

$$(A|B) := \mathrm{Tr} A^* B.$$

$B^1(\mathcal{H}_1, \mathcal{H}_2)$ denotes trace class operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, that is $A \in B^1(\mathcal{H}_1, \mathcal{H}_2)$ iff $\mathrm{Tr}(A^* A)^{1/2} < \infty$.

For a single space $\mathcal{H}$, we will write $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$, etc. $B_{\mathrm{h}}(\mathcal{H})$ will denote bounded self-adjoint operators on $\mathcal{H}$ (the subscript h stands for "hermitian"). $B_+(\mathcal{H})$ denotes positive bounded operators on $\mathcal{H}$. Similarly, $B_+^2(\mathcal{H})$ and $B_+^1(\mathcal{H})$ stand for positive Hilbert-Schmidt and trace class operators on $\mathcal{H}$ respectively.

By saying that A is an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ we will mean that the domain of A, denoted by $\mathrm{Dom}A$ is a subspace of $\mathcal{H}_1$ and A is a linear map from $\mathrm{Dom}A$ to $\mathcal{H}_2$.

The spectrum of an operator A is denoted by $\mathrm{sp}A$.

2.3 Tensor product

We assume that the reader is familiar with the concept of the algebraic tensor product of two vector spaces. The tensor product of a vector $z \in \mathcal{Z}$ and a vector $w \in \mathcal{W}$ will be, as usual, denoted by $z \otimes w$.

Let $\mathcal{Z}$ and $\mathcal{W}$ be two Hilbert spaces. The notation $\mathcal{Z} \otimes \mathcal{W}$ will be used to denote the tensor product of $\mathcal{Z}$ and $\mathcal{W}$ in the sense of Hilbert spaces. Thus $\mathcal{Z} \otimes \mathcal{W}$ is a Hilbert space equipped with the operation

$$\mathcal{Z} \times \mathcal{W} \ni (z, w) \mapsto z \otimes w \in \mathcal{Z} \otimes \mathcal{W},$$

and with a scalar product satisfying

$$(z_1 \otimes w_1 | z_2 \otimes w_2) = (z_1|z_2)(w_1|w_2). \tag{6}$$

$\mathcal{Z} \otimes \mathcal{W}$ is the completion of the algebraic tensor product of $\mathcal{Z}$ and $\mathcal{W}$ in the norm given by (6).

2.4 Operators in a tensor product

Let a and b be (not necessarily everywhere defined) operators on $\mathcal{Z}$ and $\mathcal{W}$. Then we can define a linear operator $a \otimes b$ with the domain equal to the algebraic tensor product of $\mathrm{Dom}\, a$ and $\mathrm{Dom}\, b$, satisfying

$$(a \otimes b)(z \otimes w) := (az) \otimes (bw).$$

If a and b are densely defined, then so is $a \otimes b$.

If a and b are closed, then $a \otimes b$ is closable. To see this, note that the algebraic tensor product of $\mathrm{Dom}\, a^*$ and $\mathrm{Dom}\, b^*$ is contained in the domain of $(a \otimes b)^*$. Hence $(a \otimes b)^*$ has a dense domain.

We will often denote the closure of $a \otimes b$ with the same symbol.

2.5 Conjugate Hilbert spaces

Let $\mathcal{H}$ be a complex vector space. The space $\overline{\mathcal{H}}$ conjugate to $\mathcal{H}$ is a complex vector together with a distinguished antilinear bijection

$$\mathcal{H} \ni \Psi \mapsto \overline{\Psi} \in \overline{\mathcal{H}}. \tag{7}$$

The map (7) is called a conjugation on $\mathcal{H}$. It is convenient to denote the inverse of the map (7) by the same symbol. Thus $\overline{\overline{\Psi}} = \Psi$.

Assume in addition that $\mathcal{H}$ is a Hilbert space. Then we assume that $\overline{\mathcal{H}}$ is also a Hilbert space and (7) is antiunitary, so that the scalar product on $\overline{\mathcal{H}}$ satisfies

$$(\overline{\Phi}|\overline{\Psi} = \overline{(\Phi|\Psi)}.$$

For $\Psi \in \mathcal{H}$, let $(\Psi|$ denote the operator in $B(\mathcal{H}, \mathbb{C})$ given by

$$\mathcal{H} \ni \Phi \mapsto (\Psi|\Phi) \in \mathbb{C}.$$

We will write $|\Psi) := (\Psi|^*$.

By the Riesz lemma, the map

$$\overline{\mathcal{H}} \ni \overline{\Psi} \mapsto (\Psi| \in B(\mathcal{H}, \mathbb{C})$$

is an isomorphism between $\overline{\mathcal{H}}$ and the dual of $\mathcal{H}$, that is $B(\mathcal{H}, \mathbb{C})$.

If $A \in B(\mathcal{H})$, then $\overline{A} \in B(\overline{\mathcal{H}})$ is defined by

$$\overline{\mathcal{H}} \ni \overline{\Psi} \mapsto \overline{A}\ \overline{\Psi} := \overline{A\Psi} \in \overline{\mathcal{H}}.$$

We will identify $\overline{B(\mathcal{H})}$ with $B(\overline{\mathcal{H}})$.

If $\mathcal{H}$ is a real vector space, we always take $\Psi \mapsto \overline{\Psi}$ to be the identity.

Let $\mathcal{Z}, \mathcal{W}$ be Hilbert spaces. We will often use the identification of the set of Hilbert-Schmidt operators $B^2(\mathcal{Z}, \mathcal{W})$ with $\mathcal{W} \otimes \overline{\mathcal{Z}}$, so that $|\Phi)(\Psi| \in B^2(\mathcal{Z}, \mathcal{W})$ corresponds to $\Phi \otimes \overline{\Psi} \in \mathcal{W} \otimes \overline{\mathcal{Z}}$.

We identify $\overline{\mathcal{Z} \otimes \mathcal{W}}$ with $\overline{\mathcal{W}} \otimes \overline{\mathcal{Z}}$.

If $A \in B(\mathcal{Z}, \mathcal{W})$, then $A^\# \in B(\overline{\mathcal{W}}, \overline{\mathcal{Z}})$ is defined as $A^\# := \overline{A}^*$ (recall that $*$ denotes the hermitian conjugation). $A^\#$ is sometimes called the transpose of A.

This is especially useful if $A \in B(\overline{\mathcal{Z}}, \mathcal{Z})$. Then we say that A is symmetric iff $A^\# = A$ and antisymmetric if $A^\# = -A$. In other words, A is symmetric if

$$(z_1|A\overline{z}_2) = (z_2|A\overline{z}_1), \quad z_1, z_2 \in \mathcal{Z};$$

antisymmetric if

$$(z_1|A\overline{z}_2) = -(z_2|A\overline{z}_1), \quad z_1, z_2 \in \mathcal{Z}.$$

The space of symmetric and antisymmetric bounded operators from $\overline{\mathcal{Z}}$ to $\mathcal{Z}$ is denoted by $B_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$ and $B_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$ resp. The space of Hilbert-Schmidt symmetric and antisymmetric operators from $\overline{\mathcal{Z}}$ to $\mathcal{Z}$ is denoted by $B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ and $B_{\mathrm{a}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ resp.

Remark 1. Note that, unfortunately, in the literature, e.g. in [56], the word "symmetric" is sometimes used in a different meaning ("hermitian but not necessarily self-adjoint").

2.6 Fredholm determinant

Let $\mathcal{Y}$ be a (real or complex) Hilbert space.

Let $1+B^1(\mathcal{Y})$ denote the set of operators of the form $1+a$ with $a \in B^1(\mathcal{Y})$.

Theorem 1. *There exists a unique function* $1 + B^1(\mathcal{Y}) \ni r \mapsto \det r$ *that satisfies*

1) *If* $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$ *with* $\dim \mathcal{Y}_1 < \infty$ *and* $r = r_1 \oplus 1$*, then* $\det r = \det r_1$*, where* $\det r_1$ *is the usual determinant of a finite dimensional operator* r_1*;*
2) $1 + B^1(\mathcal{Y}) \ni r \mapsto \det r$ *is continuous in the trace norm.*

$\det r$ is called the Fredholm determinant of r, see e.g. [57] Sect. XIII.17.

3 Canonical commutation relations

In this section we introduce one of the basic concept of our lectures – a representation of the canonical commutation relations (CCR). We choose the exponential form of the CCR – often called the Weyl form of the CCR.

In the literature the terminology related to CCR often depends on the author, [33, 17, 13, 25]. What we call representations of the CCR is known also as Weyl or a Heisenberg-Weyl systems.

3.1 Representations of the CCR

Let $\mathcal{Y}$ be a real vector space equipped with an antisymmetric form ω. (Note that ω does not need to be nondegenerate). Let $\mathcal{H}$ be a Hilbert space. Recall that $U(\mathcal{H})$ denotes the set of unitary operators on $\mathcal{H}$. We say that a map

$$\mathcal{Y} \ni y \mapsto W^\pi(y) \in U(\mathcal{H}) \tag{8}$$

is a *representation of the CCR over* $\mathcal{Y}$ *in* $\mathcal{H}$ if

$$W^\pi(y_1)W^\pi(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\omega y_2}W^\pi(y_1+y_2), \qquad y_1, y_2 \in \mathcal{Y}. \tag{9}$$

Note that (9) implies

Theorem 2. *Let* $y, y_1, y_2 \in \mathcal{Y}$*. Then*

$$W^\pi(y_1)W^\pi(y_2) = \mathrm{e}^{-\mathrm{i}y_1\omega y_2}W^\pi(y_2)W^\pi(y_1), \tag{10}$$

$$W^{\pi*}(y) = W^\pi(-y), \qquad W^\pi(0) = 1, \tag{11}$$

$$W^\pi(t_1 y)W^\pi(t_2 y) = W^\pi((t_1+t_2)y), \quad t_1, t_2 \in \mathbb{R}. \tag{12}$$

(10) is known as the canonical commutation relation in the Weyl form.

We say that a subset $K \subset \mathcal{H}$ is cyclic for (8) if

$$\mathrm{Span}\{W^{\pi}(y)\Psi \ : \ \Psi \in K, \ y \in \mathcal{Y}\}$$

is dense in $\mathcal{H}$. We say that $\Psi_0 \in \mathcal{H}$ is cyclic if $\{\Psi_0\}$ is cyclic.

We say that the representation (8) is irreducible if the only closed subspace of $\mathcal{H}$ preserved by $W^{\pi}(y)$ for all $y \in \mathcal{Y}$ is $\{0\}$ and $\mathcal{H}$. Clearly, in the case of an irreducible representation, all nonzero vectors in $\mathcal{H}$ are cyclic.

Suppose we are given two representations of the CCR over the same space $(\mathcal{Y}, \omega)$:

$$\mathcal{Y} \ni y \mapsto W^{\pi_1}(y) \in U(\mathcal{H}_1), \tag{13}$$

$$\mathcal{Y} \ni y \mapsto W^{\pi_2}(y) \in U(\mathcal{H}_2). \tag{14}$$

We say that (13) is unitarily equivalent to (14) iff there exists a unitary operator $U \in U(\mathcal{H}_1, \mathcal{H}_2)$ such that

$$UW^{\pi_1}(y) = W^{\pi_2}(y)U, \quad y \in \mathcal{Y}.$$

Clearly, given a representation of the CCR (8) and a linear transformation r on $\mathcal{Y}$ that preserves ω,

$$\mathcal{Y} \ni y \mapsto W^{\pi}(ry) \in U(\mathcal{H})$$

is also a representation of the CCR.

If we have two representations of the CCR

$$\mathcal{Y}_1 \ni y_1 \mapsto W^{\pi_1}(y_1) \in U(\mathcal{H}_1),$$

$$\mathcal{Y}_2 \ni y_2 \mapsto W^{\pi_2}(y_2) \in U(\mathcal{H}_2),$$

then

$$\mathcal{Y}_1 \oplus \mathcal{Y}_2 \ni (y_1, y_2) \mapsto W^{\pi_1}(y_1) \otimes W^{\pi_2}(y_2) \in U(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

is also a representation of the CCR.

By (12), for any representation of the CCR,

$$\mathbb{R} \ni t \mapsto W^{\pi}(ty) \in U(\mathcal{H}) \tag{15}$$

is a 1-parameter group. We say that a representation of the CCR (8) is *regular* if (15) is strongly continuous for each $y \in \mathcal{Y}$. Representations of the CCR that appear in applications are usually regular.

3.2 Schrödinger representation of the CCR

Let $\mathcal{X}$ be a finite dimensional real space. Let $\mathcal{X}^{\#}$ denote the space dual to $\mathcal{X}$. Then the form

$$(\eta_1, q_1)\omega(\eta_2, q_2) = \eta_1 q_2 - \eta_2 q_1 \tag{16}$$

on $\mathcal{X}^{\#} \oplus \mathcal{X}$ is symplectic.

Let x be the generic name of the variable in $\mathcal{X}$, and simultaneously the operator of multiplication by the variable x in $L^2(\mathcal{X})$. More precisely for any $\eta \in \mathcal{X}^{\#}$ the symbol ηx denotes the self-adjoint operator on $L^2(\mathcal{X})$ acting on its domain as

$$(\eta x\Psi)(x) := (\eta x)\Psi(x).$$

Let $D := \frac{1}{\mathrm{i}}\nabla_x$ be the momentum operator on $L^2(\mathcal{X})$. More precisely, for any $q \in \mathcal{X}$ the symbol qD denotes the self-adjoint operator on $L^2(\mathcal{X})$ acting on its domain as

$$(qD\Psi)(x) := \frac{1}{\mathrm{i}} q\nabla_x \Psi(x).$$

It is easy to see that

Theorem 3. *The map*

$$\mathcal{X}^{\#} \oplus \mathcal{X} \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}(\eta x + qD)} \in U(L^2(\mathcal{X})) \tag{17}$$

is an irreducible regular representation of the CCR over $\mathcal{X}^{\#} \oplus \mathcal{X}$ in $L^2(\mathcal{X})$.

(17) is called the *Schrödinger representation.*

Conversely, suppose that $\mathcal{Y}$ is a finite dimensional real vector space equipped with a symplectic form ω. Let

$$\mathcal{Y} \ni y \mapsto W^{\pi}(y) \in U(\mathcal{H}) \tag{18}$$

be a representation of the CCR. There exists a real vector space $\mathcal{X}$ such that the symplectic space $\mathcal{Y}$ can be identified with $\mathcal{X}^{\#} \oplus \mathcal{X}$ equipped with the symplectic form (16). Thus we can rewrite (18) as

$$\mathcal{X}^{\#} \oplus \mathcal{X} \ni (\eta, q) \mapsto W^{\pi}(\eta, q) \in U(\mathcal{H})$$

satisfying

$$W^{\pi}(\eta_1, q_1)W^{\pi}(\eta_2, q_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}(\eta_1 q_2 - \eta_2 q_1)} W^{\pi}(\eta_1 + \eta_2, q_1 + q_2).$$

In particular,

$$\mathcal{X}^{\#} \ni \eta \mapsto W^{\pi}(\eta, 0) \in U(\mathcal{H}), \tag{19}$$

$$\mathcal{X} \ni q \mapsto W^{\pi}(0, q) \in U(\mathcal{H}) \tag{20}$$

are unitary representations satisfying

$$W^{\pi}(\eta, 0)W^{\pi}(0, q) = \mathrm{e}^{-\mathrm{i}\eta q} W^{\pi}(0, q)W^{\pi}(\eta, 0).$$

If (18) is regular, then (19) and (20) are strongly continuous.

The following classic result says that a representation of the CCR over a symplectic space with a finite number of degrees of freedom is essentially unique up to the multiplicity (see e.g. [17, 33]):

Theorem 4 (The Stone–von Neumann theorem). *Suppose that* $(\mathcal{Y},\omega)$ *is a finite dimensional symplectic space and (18) a regular representation of the CCR. Suppose that we fix an identification of* $\mathcal{Y}$ *with* $\mathcal{X}^{\#}\oplus\mathcal{X}$*. Then, there exists a Hilbert space* $\mathcal{K}$ *and a unitary operator* $U : L^2(\mathcal{X})\otimes\mathcal{K}\to\mathcal{H}$ *such that*

$$W^\pi(\eta,q)U = U\left(\mathrm{e}^{\mathrm{i}(\eta x+qD)}\otimes 1_{\mathcal{K}}\right).$$

The representation of the CCR (18) is irreducible iff $\mathcal{K}=\mathbb{C}$.

Corollary 1. *Suppose that* $\mathcal{Y}$ *is a finite dimensional symplectic space. Let* $\mathcal{Y}\ni y\mapsto W^{\pi_1}(y)\in U(\mathcal{H})$ *and* $\mathcal{Y}\ni y\mapsto W^{\pi_2}(y)\in U(\mathcal{H})$ *be two regular irreducible representations of the CCR. Then they are unitarily equivalent.*

3.3 Field operators

In this subsection we assume that we are given a regular representation $\mathcal{Y}\ni y\mapsto W^\pi(y)\in U(\mathcal{H})$. Recall that $\mathbb{R}\ni t\mapsto W^\pi(ty)$ is a strongly continuous unitary group. By the Stone theorem, for any $y\in\mathcal{Y}$, we can define its self-adjoint generator

$$\phi^\pi(y) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}W^\pi(ty)\Big|_{t=0}.$$

$\phi^\pi(y)$ will be called the *field operator* corresponding to $y\in\mathcal{Y}$.

Remark 2. Sometimes, the operators $\phi^\pi(y)$ are called *Segal field operators.*

Theorem 5. *Let* $y, y_1, y_2\in\mathcal{Y}$.

1) *Let* $\Psi\in\mathrm{Dom}\,\phi^\pi(y_1)\cap\mathrm{Dom}\,\phi^\pi(y_2)$*,* $c_1,c_2\in\mathbb{R}$*. Then*

$$\Psi\in\mathrm{Dom}\,\phi^\pi(c_1y_1+c_2y_2),$$

$$\phi^\pi(c_1y_1+c_2y_2)\Psi = c_1\phi^\pi(y_1)\Psi + c_2\phi^\pi(y_2)\Psi.$$

2) *Let* $\Psi_1,\Psi_2\in\mathrm{Dom}\;\phi^\pi(y_1)\cap\mathrm{Dom}\;\phi^\pi(y_2)$*. Then*

$$(\phi^\pi(y_1)\Psi_1|\phi^\pi(y_2)\Psi_2) - (\phi^\pi(y_2)\Psi_1|\phi^\pi(y_1)\Psi_2) = \mathrm{i}y_1\omega y_2(\Psi_1|\Psi_2). \qquad (21)$$

3) $\phi^\pi(y_1)+\mathrm{i}\phi^\pi(y_2)$ *is a closed operator on the domain*

$$\mathrm{Dom}\;\phi^\pi(y_1)\cap\mathrm{Dom}\;\phi^\pi(y_2).$$

(21) can be written somewhat imprecisely as

$$[\phi^\pi(y_1),\phi^\pi(y_2)] = \mathrm{i}y_1\omega y_2 \qquad (22)$$

and is called the canonical commutation relation in the Heisenberg form.

3.4 Bosonic Bogolubov transformations

Let $(\mathcal{Y}, \omega)$ be a finite dimensional symplectic space. Linear maps on $\mathcal{Y}$ preserving ω are then automatically invertible. They form a group, which will be called the *symplectic group of* $\mathcal{Y}$ and denoted $Sp(\mathcal{Y})$.

Let $\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H})$ be a regular irreducible representation of canonical commutation relations. The following theorem is an immediate consequence of Corollary 1:

Theorem 6. *For any $r \in Sp(\mathcal{Y})$ there exists $U \in U(\mathcal{H})$, defined uniquely up to a phase factor (a complex number of absolute value* 1*), such that*

$$UW(y)U^* = W(ry). \tag{23}$$

Let $\mathcal{U}_r$ be the class of unitary operators satisfying (23). Then

$$Sp(\mathcal{Y}) \ni r \mapsto \mathcal{U}_r \in U(\mathcal{H})/U(1)$$

is a group homomorphism, where $U(1)$ denotes the group of unitary scalar operators on $\mathcal{H}$.

One can ask whether one can fix uniquely the phase factor appearing in the above theorem and obtain a group homomorphism of $Sp(\mathcal{Y})$ into $U(\mathcal{H})$ satisfying (23). This is impossible, the best what one can do is the following improvement of Theorem 6

Theorem 7. *For any $r \in Sp(\mathcal{Y})$ there exists a unique pair of operators $\{U_r, -U_r\} \subset U(\mathcal{H})$ such that*

$$U_r W(y) U_r^* = W(ry),$$

and such that we have a group homomorphism

$$Sp(\mathcal{Y}) \ni r \mapsto \pm U_r \in U(\mathcal{H})/\{1, -1\}. \tag{24}$$

The representation (24) is called the *metaplectic representation of* $Sp(\mathcal{Y})$.

Note that the homotopy group of $Sp(\mathcal{Y})$ is $\mathbb{Z}$. Hence for any $n \in \{1, 2, \dots, \infty\}$ we can construct the n-fold covering of $Sp(\mathcal{Y})$. The image of the metaplectic representation is isomorphic to the double covering of $Sp(\mathcal{Y})$. It is often called the metaplectic group.

In the physics literature the fact that symplectic transformations can be unitarily implemented is generally associated with the name of Bogolubov, who successfully applied this idea to the superfluidity of the Bose gas. Therefore, in the physics literature, the transformations described in Theorems 6 and 7 are often called *Bogolubov transformations*.

The proofs of Theorems 6 and 7 are most conveniently given in the Fock representation, where one has simple formulas for U_r (see e.g. [36, 14]). We will describe these formulas later on (in the more general context of an infinite number of degrees of freedom).

4 Canonical anticommutation relations

In this section we introduce the second basic concept of our lectures, that of a representation of the canonical anticommutation relations (CAR). Again, there is no uniform terminology in this domain [33, 17, 13]. What we call a representation of the CAR is often called Clifford relations, which is perhaps more justified historically. Our terminology is intended to stress the analogy between the CCR and CAR.

4.1 Representations of the CAR

Let $\mathcal{Y}$ be a real vector space with a positive scalar product α. Let $\mathcal{H}$ be a Hilbert space. Recall that $B_{\mathrm{h}}(\mathcal{H})$ denotes the set of bounded self-adjoint operators on $\mathcal{H}$. We say that a linear map

$$\mathcal{Y} \ni y \mapsto \phi^\pi(y) \in B_{\mathrm{h}}(\mathcal{H}) \tag{25}$$

is a *representation of the CAR over $\mathcal{Y}$ in $\mathcal{H}$* iff $\phi^\pi(y)$ satisfy

$$[\phi^\pi(y_1), \phi^\pi(y_2)]_+ = 2y_1\alpha y_2, \quad y_1, y_2 \in \mathcal{Y}. \tag{26}$$

We will often drop π, and write $|y|_\alpha := (y\alpha y)^{1/2}$.

Remark 3. The reason of putting the factor 2 in (26) is the identity $\phi(y)^2 = y\alpha y$. Note, however, that in (22) there is no factor of 2, and therefore at some places the treatment of the CCR and CAR will be not as parallel as it could be.

Theorem 8. 1) sp $\phi(y) = \{-|y|_\alpha, |y|_\alpha\}$
2) *Let $t \in \mathbb{C}$, $y \in \mathcal{Y}$. Then* $\|t + \phi(y)\| = \max\{|t + |y|_\alpha|, |t - |y|_\alpha|\}$.
3) $\mathrm{e}^{\mathrm{i}\phi(y)} = \cos|y|_\alpha + \mathrm{i}\frac{\sin|y|_\alpha}{|y|_\alpha}\phi(y)$.
4) *Let $\mathcal{Y}^{\mathrm{cpl}}$ be the completion of $\mathcal{Y}$ in the norm $|\cdot|_\alpha$. Then there exists a unique extension of (25) to a continuous map*

$$\mathcal{Y}^{\mathrm{cpl}} \ni y \mapsto \phi^{\pi^{\mathrm{cpl}}}(y) \in B_{\mathrm{h}}(\mathcal{H}). \tag{27}$$

Moreover, (27) is a representation of the CAR.

Motivated by the last statement, henceforth we will assume that $\mathcal{Y}$ is complete—that is, $\mathcal{Y}$ is a real Hilbert space.

By saying that $(\phi_1, \dots, \phi_n)$ is a representation of the CAR on $\mathcal{H}$ we will mean that we have a representation of the CAR $\mathbb{R}^n \ni y \mapsto \phi(y) \in B_{\mathrm{h}}(\mathcal{H})$, where $\mathbb{R}^n$ is equipped with the canonical scalar product, e_i is the canonical basis of $\mathbb{R}^n$ and $\phi_i = \phi^\pi(e_i)$. Clearly, this is equivalent to the relations $[\phi_i, \phi_j]_+ = 2\delta_{ij}$.

We say that a subset $K \subset \mathcal{H}$ is cyclic for (25) if $\mathrm{Span}\{\phi^\pi(y_1)\cdots\phi^\pi(y_n)\Psi\ :\ \Psi \in K,\ y_1,\dots,y_n \in \mathcal{Y},\ n=1,2,\dots\}$ is dense in $\mathcal{H}$. We say that $\Psi_0 \in \mathcal{H}$ is cyclic for (25) if $\{\Psi_0\}$ is cyclic.

We say that (25) is irreducible if the only closed subspace of $\mathcal{H}$ preserved by $\phi^\pi(y)$ for all $y \in \mathcal{Y}$ is $\{0\}$ and $\mathcal{H}$. Clearly, in the case of an irreducible representation, all nonzero vectors in $\mathcal{H}$ are cyclic.

Suppose we are given two representations of the CAR over the same space $(\mathcal{Y},\alpha)$:

$$\mathcal{Y} \ni y \mapsto \phi^{\pi_1}(y) \in B_\mathrm{h}(\mathcal{H}_1), \tag{28}$$

$$\mathcal{Y} \ni y \mapsto \phi^{\pi_2}(y) \in B_\mathrm{h}(\mathcal{H}_2), \tag{29}$$

then we say that (28) is unitarily equivalent to (29) iff there exists a unitary operator $U \in U(\mathcal{H}_1,\mathcal{H}_2)$ such that

$$U\phi^{\pi_1}(y) = \phi^{\pi_2}(y)U, \quad y \in \mathcal{Y}.$$

Let $\mathcal{Y}_1$, $\mathcal{Y}_2$ be two real Hilbert spaces. Suppose that I is a self-adjoint operator on $\mathcal{H}_1$ and

$$\mathcal{Y}_1 \oplus \mathbb{R} \ni (y_1,t) \mapsto \phi^{\pi_1}(y_1) + tI \in B_\mathrm{h}(\mathcal{H}_1),$$

$$\mathcal{Y}_2 \ni y_2 \mapsto \phi^{\pi_2}(y_2) \in B_\mathrm{h}(\mathcal{H}_2)$$

are representations of the CAR. Then

$$\mathcal{Y}_1 \oplus \mathcal{Y}_2 \ni (y_1,y_2) \mapsto \phi^{\pi_1}(y_1)\otimes 1 + I \otimes \phi^{\pi_2}(y_2) \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

is a representation of the CAR.

If $r \in B(\mathcal{Y})$ preserves the scalar product (is isometric), and we are given a representation of the CAR (25), then

$$\mathcal{Y} \ni y \mapsto \phi^\pi(ry) \in B_\mathrm{h}(\mathcal{H})$$

is also a representation of the CAR.

Most of the above material was very similar to its CCR counterpart. The following construction, however, has no analog in the context of the CCR:

Theorem 9. *Suppose that $\mathbb{R}^n \ni y \mapsto \phi(y)$ is a representation of the CAR. Let $y_1,\dots,y_n$ be an orthonormal basis in $\mathbb{R}^n$. Set*

$$Q := \mathrm{i}^{n(n-1)/2}\phi(y_1)\cdots\phi(y_n).$$

Then the following is true:

1) *Q depends only on the orientation of the basis (it changes the sign under the change of the orientation).*
2) *Q is unitary and self-adjoint, moreover, $Q^2 = 1$.*

3) *$Q\phi(y) = (-1)^n \phi(y) Q$, for any $y \in \mathcal{Y}$.*
4) *If $n = 2m$, then $Q = \mathrm{i}^m \phi(y_1) \cdots \phi(y_{2m})$ and*

$$\mathbb{R}^{2m+1} \ni (y,t) \mapsto \phi(y) \pm tQ$$

are two representations of the CAR.
5) *If $n = 2m+1$, then $Q = (-\mathrm{i})^m \phi(y_1) \cdots \phi(y_{2m+1})$ and $\mathcal{H} = \mathrm{Ker}(Q-1) \oplus \mathrm{Ker}(Q+1)$ gives a decomposition of $\mathcal{H}$ into a direct sum of subspaces preserved by our representation.*

4.2 Representations of the CAR in terms of Pauli matrices

In the space $\mathbb{C}^2$ we introduce the usual Pauli spin matrices σ_1, σ_2 and σ_3. This means

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that $\sigma_i^2 = 1$, $\sigma_i^* = \sigma_i$, $i = 1,2,3$, and

$$\begin{aligned} \sigma_1\sigma_2 &= -\sigma_2\sigma_1 = \mathrm{i}\sigma_3, \\ \sigma_2\sigma_3 &= -\sigma_3\sigma_2 = \mathrm{i}\sigma_1 \\ \sigma_3\sigma_1 &= -\sigma_1\sigma_3 = \mathrm{i}\sigma_2. \end{aligned} \tag{30}$$

Moreover, $B(\mathbb{C}^2)$ has a basis $(1, \sigma_1, \sigma_2, \sigma_3)$. Clearly, $(\sigma_1, \sigma_2, \sigma_3)$ is a representation of the CAR over $\mathbb{R}^3$.

In the algebra $B(\otimes^n \mathbb{C}^2)$ we introduce the operators

$$\sigma_i^{(j)} := 1^{\otimes(j-1)} \otimes \sigma_i \otimes 1^{\otimes(n-j)}, \quad i = 1,2,3, \quad j = 1,\dots,n.$$

Note that $\sigma_i^{(j)}$ satisfy (30) for any j and commute for distinct j. Moreover, $B(\otimes^n \mathbb{C}^2)$ is generated as an algebra by $\{\sigma_i^{(j)} \ : \ j = 1,\dots,n, \ i = 1,2\}$. Set $I_j := \sigma_3^{(1)} \cdots \sigma_3^{(j)}$. In order to transform spin matrices into a representation of the CAR we need to apply the so-called *Jordan-Wigner construction*. According to this construction,

$$\left(\sigma_1^{(1)}, \sigma_2^{(1)}, I_1\sigma_1^{(2)}, I_1\sigma_2^{(2)}, \dots, I_{n-1}\sigma_1^{(n)}, I_{n-1}\sigma_2^{(n)}\right)$$

is a representation of the CAR over $\mathbb{R}^{2n}$. By adding $\pm I_n$ we obtain a representation of the CAR over $\mathbb{R}^{2n+1}$.

The following theorem can be viewed as a fermionic analog of the Stone-von Neumann Theorem 4. It is, however, much easier to prove and belongs to standard results about Clifford algebras [47].

Theorem 10. *1) Let $(\phi_1, \phi_2, \dots, \phi_{2n})$ be a representation of the CAR over $\mathbb{R}^{2n}$ in a Hilbert space $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K}$ and a unitary operator*

$$U : \otimes^n \mathbb{C}^2 \otimes \mathcal{K} \to \mathcal{H}$$

such that

$$U\left(I_{j-1}\sigma_1^{(j)} \otimes 1_{\mathcal{K}}\right) = \phi_{2j-1}U,$$
$$U\left(I_{j-1}\sigma_2^{(j)} \otimes 1_{\mathcal{K}}\right) = \phi_{2j}U, \quad j = 1, \dots, n.$$

The representation is irreducible iff $\mathcal{K} = \mathbb{C}$.

2) Let $(\phi_1, \phi_2, \dots, \phi_{2n+1})$ be a representation of the CAR over $\mathbb{R}^{2n+1}$ in a Hilbert space $\mathcal{H}$. Then there exist Hilbert spaces $\mathcal{K}_-$ and $\mathcal{K}_+$ and a unitary operator

$$U : \otimes^n \mathbb{C}^2 \otimes (\mathcal{K}_+ \oplus \mathcal{K}_-) \to \mathcal{H}$$

such that

$$U\left(I_{j-1}\sigma_1^{(j)} \otimes 1_{\mathcal{K}_+ \oplus \mathcal{K}_-}\right) = \phi_{2j-1}U,$$
$$U\left(I_{j-1}\sigma_2^{(j)} \otimes 1_{\mathcal{K}_+ \oplus \mathcal{K}_-}\right) = \phi_{2j}U, \quad j = 1, \dots, n,$$
$$U\left(I_n \otimes (1_{\mathcal{K}_+} \oplus -1_{\mathcal{K}_-})\right) = \phi_{2n+1}U.$$

Corollary 2. *Suppose that $\mathcal{Y}$ is an even dimensional real Hilbert space. Let $\mathcal{Y} \ni y \mapsto \phi^{\pi_1}(y) \in B_{\mathrm{h}}(\mathcal{H})$ and $\mathcal{Y} \ni y \mapsto \phi^{\pi_2}(y) \in B_{\mathrm{h}}(\mathcal{H})$ be two irreducible representations of the CAR. Then they are unitarily equivalent.*

4.3 Fermionic Bogolubov transformations

Let $(\mathcal{Y}, \alpha)$ be a finite dimensional real Hilbert space (a Euclidean space). Linear transformations on $\mathcal{Y}$ that preserve the scalar product are invertible and form a group, which will be called the orthogonal group of $\mathcal{Y}$ and denoted $O(\mathcal{Y})$.

Let

$$\mathcal{Y} \ni y \mapsto \phi(y) \in B_{\mathrm{h}}(\mathcal{H}) \tag{31}$$

be an irreducible representation of the CAR. The following theorem is an immediate consequence of Corollary 2:

Theorem 11. *Let $\dim \mathcal{Y}$ be even. For any $r \in O(\mathcal{Y})$ there exists $U \in U(\mathcal{H})$ such that*

$$U\phi(y)U^* = \phi(ry). \tag{32}$$

The unitary operator U in (32) is defined uniquely up to a phase factor. Let $\mathcal{U}_r$ denote the class of such operators. Then

$$O(\mathcal{Y}) \ni r \mapsto \mathcal{U}_r \in U(\mathcal{H})/U(1)$$

is a group homomorphism.

One can ask whether one can fix uniquely the phase factor appearing in the above theorem and obtain a group homomorphism of $O(\mathcal{Y})$ into $U(\mathcal{H})$ satisfying (32). This is impossible, the best one can do is the following improvement of Theorem 11:

Theorem 12. *Let* $\dim\mathcal{Y}$ *be even. For any* $r \in O(\mathcal{Y})$ *there exists a unique pair* $\{U_r, -U_r\} \subset U(\mathcal{H})$ *such that*

$$U_r\phi(y)U_r^* = \phi(ry),$$

and such that we have a group homomorphism

$$O(\mathcal{Y}) \ni r \mapsto \pm U_r \in U(\mathcal{H})/\{1,-1\}. \tag{33}$$

(33) is called the *Pin representation of* $O(\mathcal{Y})$.

Note that for $\dim\mathcal{Y} > 2$, the homotopy group of $O(\mathcal{Y})$ is $\mathbb{Z}_2$. Hence the double covering of $O(\mathcal{Y})$ is its universal covering. The image of the Pin representation in $U(\mathcal{H})$ is isomorphic to this double covering and is called the Pin group.

In the physics literature the fact that orthogonal transformations can be unitarily implemented is again associated with the name of Bogolubov and the transformations described in Theorems 11 and 12 are often called (fermionic) Bogolubov transformations. They are used e.g. in the theory of the superconductivity.

Theorems 11 and 12 are well known in mathematics in the context of theory of Clifford algebras. They are most conveniently proven by using, what we call, the Fock representation, where one has simple formulas for U_r. We will describe these formulas later on (in a more general context of the infinite number of degrees of freedom).

5 Fock spaces

In this section we fix the notation for bosonic and fermionic Fock spaces. Even though these concepts are widely used, there seem to be no universally accepted symbols for many concepts in this area.

5.1 Tensor algebra

Let $\mathcal{Z}$ be a Hilbert space. Let $\otimes^n\mathcal{Z}$ denote the n-fold tensor product of $\mathcal{Z}$. We set

$$\otimes\mathcal{Z} = \mathop{\oplus}_{n=0}^{\infty} \otimes^n\mathcal{Z}.$$

Here $\oplus$ denotes the direct sum in the sense of Hilbert spaces, that is the completion of the algebraic direct sum. $\otimes\mathcal{Z}$ is sometimes called the *full Fock*

space. The element $1 \in \mathbb{C} = \otimes^0 \mathcal{Z} \subset \otimes \mathcal{Z}$ is often called the *vacuum* and denoted Ω.

Sometimes we will need $\otimes^{\mathrm{fin}} \mathcal{Z}$ which is the subspace of $\otimes \mathcal{Z}$ with a finite number of particles, that means the algebraic direct sum of $\otimes^n \mathcal{Z}$.

$\otimes \mathcal{Z}$ and $\otimes^{\mathrm{fin}} \mathcal{Z}$ are associative algebras with the operation $\otimes$ and the identity Ω.

5.2 Operators $\mathrm{d}\Gamma$ and Γ in the full Fock space

If p is a closed operator from $\mathcal{Z}_1$ to $\mathcal{Z}_2$, then we define the closed operator $\Gamma^n(p)$ from $\otimes^n \mathcal{Z}_1$ to $\otimes^n \mathcal{Z}_2$ and $\Gamma(p)$ from $\otimes \mathcal{Z}_1$ to $\otimes \mathcal{Z}_2$:

$$\Gamma^n(p) := p^{\otimes n},$$

$$\Gamma(p) := \mathop{\oplus}_{n=0}^{\infty} \Gamma^n(p).$$

$\Gamma(p)$ is bounded iff $\|p\| \leq 1$. $\Gamma(p)$ is unitary iff p is.

Likewise, if h is a closed operator on $\mathcal{Z}$, then we define the closed operator $\mathrm{d}\Gamma^n(h)$ on $\otimes^n \mathcal{Z}$ and $\mathrm{d}\Gamma(h)$ on $\otimes \mathcal{Z}$:

$$\mathrm{d}\Gamma^n(h) = \sum_{j=1}^{n} 1_{\mathcal{Z}}^{\otimes(j-1)} \otimes h \otimes 1_{\mathcal{Z}}^{\otimes(n-j)},$$

$$\mathrm{d}\Gamma(h) := \mathop{\oplus}_{n=0}^{\infty} \mathrm{d}\Gamma^n(h).$$

$\mathrm{d}\Gamma(h)$ is self-adjoint iff h is.

The *number operator* is defined as $N = \mathrm{d}\Gamma(1)$. The *parity operator* is

$$I := (-1)^N = \Gamma(-1). \tag{34}$$

Let us give a sample of properties of operators on full Fock spaces.

Theorem 13. 1) *Let $h, h_1, h_2 \in B(\mathcal{Z})$, $p_1 \in B(\mathcal{Z}, \mathcal{Z}_1)$, $p_2 \in B(\mathcal{Z}_1, \mathcal{Z}_2)$, $\|p_i\| \leq 1$. We then have*

$$\Gamma(\mathrm{e}^{\mathrm{i}h}) = \exp(\mathrm{d}\Gamma(\mathrm{i}h)), \tag{35}$$

$$\Gamma(p_2)\Gamma(p_1) = \Gamma(p_2 p_1),$$

$$[\mathrm{d}\Gamma(h_1), \mathrm{d}\Gamma(h_2)] = \mathrm{d}\Gamma([h_1, h_2]).$$

2) *Let $\Phi, \Psi \in \otimes^{\mathrm{fin}} \mathcal{Z}$, $h \in B(\mathcal{Z})$, $p \in B(\mathcal{Z}, \mathcal{Z}_1)$. Then*

$$\Gamma(p)\ (\Phi \otimes \Psi) = (\Gamma(p)\Phi) \otimes (\Gamma(p)\Psi),$$

$$\mathrm{d}\Gamma(h)\ (\Phi \otimes \Psi) = (\mathrm{d}\Gamma(h)\Phi) \otimes \Psi + \Phi \otimes (\mathrm{d}\Gamma(h)\Psi).$$

Of course, under appropriate technical conditions, similar statements are true for unbounded operators. In particular (35) is true for any self-adjoint h.

5.3 Symmetric and antisymmetric Fock spaces

Let $S^n \ni \sigma \mapsto \Theta(\sigma) \in U(\otimes^n \mathcal{Z})$ be the natural representation of the permutation group S^n given by

$$\Theta(\sigma) z_1 \otimes \cdots \otimes z_n := z_{\sigma^{-1}(1)} \otimes \cdots \otimes z_{\sigma^{-1}(n)}.$$

We define

$$\Theta_{\mathrm{s}}^n := \frac{1}{n!} \sum_{\sigma \in S^n} \Theta(\sigma),$$

$$\Theta_{\mathrm{a}}^n := \frac{1}{n!} \sum_{\sigma \in S^n} (\operatorname{sgn} \sigma)\, \Theta(\sigma).$$

Θ_{s}^n and Θ_{a}^n are orthogonal projections in $\otimes^n \mathcal{Z}$.

We will write s/a as a subscript that can mean either s or a. We set

$$\Theta_{\mathrm{s/a}} := \mathop{\oplus}_{n=0}^{\infty} \Theta_{\mathrm{s/a}}^n.$$

Clearly, $\Theta_{\mathrm{s/a}}$ is an orthogonal projection in $\otimes \mathcal{Z}$.

Define

$$\Gamma_{\mathrm{s/a}}^n(\mathcal{Z}) := \Theta_{\mathrm{s/a}}^n \big(\otimes^n \mathcal{Z}\big),$$

$$\Gamma_{\mathrm{s/a}}(\mathcal{Z}) := \Theta_{\mathrm{s/a}} \big(\otimes \mathcal{Z}\big) = \mathop{\oplus}_{n=0}^{\infty} \Gamma_{\mathrm{s/a}}^n(\mathcal{Z}).$$

$\Gamma_{\mathrm{s/a}}(\mathcal{Z})$ is often called the *bosonic/fermionic* or *symmetric/antisymmetric Fock space*.

We also introduce the finite particle Fock spaces

$$\Gamma_{\mathrm{s/a}}^{\mathrm{fin}}(\mathcal{Z}) = \big(\otimes^{\mathrm{fin}} \mathcal{Z}\big) \cap \Gamma_{\mathrm{s/a}}(\mathcal{Z}).$$

$\Gamma_{\mathrm{s/a}}(\mathcal{Z})$ is a Hilbert space (as a closed subspace of $\otimes \mathcal{Z}$).

Note that $\Gamma_{\mathrm{s/a}}^0(\mathcal{Z}) = \mathbb{C}$ and $\Gamma_{\mathrm{s/a}}^1(\mathcal{Z}) = \mathcal{Z}$. $\mathcal{Z}$ is often called the *1-particle space* and $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$ the second quantization of $\mathcal{Z}$.

The following property of bosonic Fock spaces is often useful:

Theorem 14. *The span of elements of the form $z^{\otimes n}$, $z \in \mathcal{Z}$, is dense in $\Gamma_{\mathrm{s}}^n(\mathcal{Z})$.*

5.4 Symmetric and antisymmetric tensor product

If $\Psi, \Phi \in \Gamma_{\mathrm{s/a}}^{\mathrm{fin}}(\mathcal{Z})$, then we define

$$\Psi \otimes_{\mathrm{s/a}} \Phi := \Theta_{\mathrm{s/a}} \big(\Psi \otimes \Phi\big) \in \Gamma_{\mathrm{s/a}}^{\mathrm{fin}}(\mathcal{Z}).$$

$\Gamma_{\mathrm{s/a}}^{\mathrm{fin}}(\mathcal{Z})$ is an associative algebra with the operation $\otimes_{\mathrm{s/a}}$ and the identity Ω.

Note that $z^{\otimes n} = z^{\otimes_{\mathrm{s}} n}$.

Instead of $\otimes_{\rm a}$ one often uses the wedge product, which for $\Psi \in \Gamma_{\rm a}^p(\mathcal{Z})$, $\Phi \in \Gamma_{\rm a}^r(\mathcal{Z})$ is defined as

$$\Psi \wedge \Phi := \frac{(p+r)!}{p!r!} \Psi \otimes_{\rm a} \Phi \in \Gamma_{\rm a}^{\rm fin}(\mathcal{Z}).$$

It is also an associative operation. Its advantage over $\otimes_{\rm a}$ is visible if we compare the following identities:

$$\begin{aligned} v_1 \wedge \cdots \wedge v_p &= \sum_{\sigma \in S^p} (\mathrm{sgn}\sigma)\ v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}, \\ v_1 \otimes_{\rm a} \cdots \otimes_{\rm a} v_p &= \tfrac{1}{p!} \sum_{\sigma \in S^p} (\mathrm{sgn}\sigma)\ v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}, \quad v_1, \cdots, v_p \in \mathcal{Z}. \end{aligned}$$

The advantage of $\otimes_{\rm a}$ is that it is fully analogous to $\otimes_{\rm s}$.

5.5 dΓ and Γ operations

If p is a closed operator from $\mathcal{Z}$ to $\mathcal{W}$, then $\Gamma^n(p)$, defined in Subsect. 5.2, maps $\Gamma_{\rm s/a}^n(\mathcal{Z})$ into $\Gamma_{\rm s/a}^n(\mathcal{W})$. Hence $\Gamma(p)$ maps $\Gamma_{\rm s/a}(\mathcal{Z})$ into $\Gamma_{\rm s/a}(\mathcal{W})$. We will use the same symbols $\Gamma^n(p)$ and $\Gamma(p)$ to denote the corresponding restricted operators.

If h is a closed operator on $\mathcal{Z}$, then $\mathrm{d}\Gamma^n(h)$ maps $\Gamma_{\rm s/a}^n(\mathcal{Z})$ into itself. Hence, $\mathrm{d}\Gamma(h)$ maps $\Gamma_{\rm s/a}(\mathcal{Z})$ into itself. We will use the same symbols $\mathrm{d}\Gamma^n(h)$ and $\mathrm{d}\Gamma(h)$ to denote the corresponding restricted operators.

$\Gamma(p)$ is called the 2nd quantization of p. Similarly, $\mathrm{d}\Gamma(h)$ is sometimes called the 2nd quantization of h.

Theorem 15. *Let $p \in B(\mathcal{Z}, \mathcal{Z}_1)$, $h \in B(\mathcal{Z})$, $\Psi, \Phi \in \Gamma_{\rm s/a}^{\rm fin}(\mathcal{Z})$. Then*

$$\begin{aligned} \Gamma(p)\ \big(\Psi \otimes_{\rm s/a} \Phi\big) &= (\Gamma(p)\Psi) \otimes_{\rm s/a} (\Gamma(p)\Phi), \\ \mathrm{d}\Gamma(h)\ \big(\Psi \otimes_{\rm s/a} \Phi\big) &= (\mathrm{d}\Gamma(h)\Psi) \otimes_{\rm s/a} \Phi + \Psi \otimes_{\rm s/a} (\mathrm{d}\Gamma(h)\Phi). \end{aligned}$$

5.6 Tensor product of Fock spaces

In this subsection we describe the so-called *exponential law for Fock spaces.*

Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be Hilbert spaces. We introduce the identification

$$U : \Gamma_{\rm s/a}^{\rm fin}(\mathcal{Z}_1) \otimes \Gamma_{\rm s/a}^{\rm fin}(\mathcal{Z}_2) \to \Gamma_{\rm s/a}^{\rm fin}(\mathcal{Z}_1 \oplus \mathcal{Z}_2)$$

as follows. Let $\Psi_1 \in \Gamma_{\rm s/a}^n(\mathcal{Z}_1)$, $\Psi_2 \in \Gamma_{\rm s/a}^m(\mathcal{Z}_2)$. Let j_i be the imbedding of $\mathcal{Z}_i$ in $\mathcal{Z}_1 \oplus \mathcal{Z}_2$. Then

$$U(\Psi_1 \otimes \Psi_2) := \sqrt{\tfrac{(n+m)!}{n!m!}} (\Gamma(j_1)\Psi_1) \otimes_{\rm s/a} (\Gamma(j_2)\Psi_2)\,. \tag{36}$$

Theorem 16. 1) *$U(\Omega_1 \otimes \Omega_2) = \Omega$.*

2) *U extends to a unitary operator* $\Gamma_{\mathrm{s/a}}(\mathcal{Z}_1)\otimes\Gamma_{\mathrm{s/a}}(\mathcal{Z}_2)\to\Gamma_{\mathrm{s/a}}(\mathcal{Z}_1\oplus\mathcal{Z}_2)$.
3) *If* $h_i\in B(\mathcal{Z}_i)$, *then*

$$U\big(\mathrm{d}\Gamma(h_1)\otimes 1+1\otimes\mathrm{d}\Gamma(h_2)\big)=\mathrm{d}\Gamma(h_1\oplus h_2)U.$$

4) *If* $p_i\in B(\mathcal{Z}_i)$, *then*

$$U\left(\Gamma(p_1)\otimes\Gamma(p_2)\right)=\Gamma(p_1\oplus p_2)U.$$

5.7 Creation and annihilation operators

Let $\mathcal{Z}$ be a Hilbert space and $w\in\mathcal{Z}$. We consider the bosonic/fermionic Fock space $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$.

Let $w\in\mathcal{Z}$. We define two operators with the domain $\Gamma_{\mathrm{s/a}}^{\mathrm{fin}}(\mathcal{Z})$. The *creation operator* is defined as

$$a^*(w)\Psi:=\sqrt{n+1}w\otimes_{\mathrm{s/a}}\Psi,\quad \Psi\in\Gamma_{\mathrm{s/a}}^n(\mathcal{Z})$$

In the fermionic case, $a^*(w)$ is bounded. In the bosonic case, $a^*(w)$ is densely defined and closable. In both cases we define denote the closure of $a^*(w)$ by the same symbol. Likewise, in both cases we define the *annihilation operator* by

$$a(w):=a^*(w)^*.$$

Note that

$$a(w)\Psi=\sqrt{n}\big((w|\otimes 1\big)\Psi,\quad \Psi\in\Gamma_{\mathrm{s/a}}^n(\mathcal{Z}).$$

Theorem 17. 1) *In the bosonic case we have*

$$[a^*(w_1),a^*(w_2)]=0,\quad [a(w_1),a(w_2)]=0,$$
$$[a(w_1),a^*(w_2)]=(w_1|w_2).$$

2) *In the fermionic case we have*

$$[a^*(w_1),a^*(w_2)]_+=0,\quad [a(w_1),a(w_2)]_+=0,$$
$$[a(w_1),a^*(w_2)]_+=(w_1|w_2).$$

In both bosonic and fermionic cases the following is true:

Theorem 18. *Let* $p,h\in B(\mathcal{Z})$ *and* $w\in\mathcal{Z}$. *Then*
1) $\Gamma(p)a(w)=a(p^{*-1}w)\Gamma(p)$,
2) $[\mathrm{d}\Gamma(h),a(w)]=-a(h^*w)$,
3) $\Gamma(p)a^*(w)=a^*(pw)\Gamma(p)$,
4) $[\mathrm{d}\Gamma(h),a^*(w)]=a^*(hw)$.

The exponential law for creation/annihilation operators is slightly different in the bosonic and fermionic cases:

Theorem 19. *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be Hilbert spaces and $(w_1, w_2) \in \mathcal{Z}_1 \oplus \mathcal{Z}_2$. Let U be the defined for these spaces as in Theorem 16.*

1) *In the bosonic case we have*

$$a^*(w_1, w_2)U = U(a^*(w_1) \otimes 1 + 1 \otimes a^*(w_2)),$$

$$a(w_1, w_2)U = U(a(w_1) \otimes 1 + 1 \otimes a(w_2)).$$

2) *In the fermionic case, if I_1 denotes the parity operator for $\Gamma_{\mathrm{a}}(\mathcal{Z}_1)$ (see (34)), then*

$$a^*(w_1, w_2)U = U(a^*(w_1) \otimes 1 + I_1 \otimes a^*(w_2)),$$

$$a(w_1, w_2)U = U(a(w_1) \otimes 1 + I_1 \otimes a(w_2)).$$

Set

$$\Lambda := (-1)^{N(N-1)/2}. \tag{37}$$

The following property is valid both in the bosonic and fermionic cases:

$$\begin{aligned} \Lambda a^*(z)\Lambda &= -Ia^*(z) = a^*(z)I, \\ \Lambda a(z)\Lambda &= Ia(z) = -a(z)I. \end{aligned} \tag{38}$$

In the fermionic case, (38) allows to convert the anticommutation relations into commutation relations

$$[\Lambda a^*(z)\Lambda, a^*(w)] = [\Lambda a(z)\Lambda, a(w)] = 0,$$

$$[\Lambda a^*(z)\Lambda, a(w)] = I(w|z).$$

Theorem 20. *Let the assumptions of Theorem 19 be satisfied. Let N_i, I_i, Λ_i be the operators defined as above corresponding to $\mathcal{Z}_i$, $i = 1, 2$. Then*

1) $\Lambda U = U(\Lambda_1 \otimes \Lambda_2)(-1)^{N_1 \otimes N_2}$.
2) *In the fermionic case,*

$$\Lambda a^*(w_1, w_2)\Lambda\, U = U(a^*(w_1)I_1 \otimes I_2 + 1 \otimes a^*(w_2)I_2),$$

$$\Lambda a(w_1, w_2)\Lambda\, U = U(-a(w_1)I_1 \otimes I_2 - 1 \otimes a(w_2)I_2).$$

Proof. To prove 2) we use 1) and $(-1)^{N_1 \otimes N_2} a(w) \otimes 1 (-1)^{N_1 \otimes N_2} = a(w) \otimes I_2$.
□

5.8 Multiple creation and annihilation operators

Let $\Phi \in \Gamma^m_{\rm s/a}(\mathcal{Z})$. We define the operator of creation of Φ with the domain $\Gamma^{\rm fin}_{\rm s/a}(\mathcal{Z})$ as follows:

$$a^*(\Phi)\Psi := \sqrt{(n+1)\cdots(n+m)}\Phi \otimes_{\rm s/a} \Psi, \quad \Psi \in \Gamma^n_{\rm s/a}.$$

$a^*(\Phi)$ is a densely defined closable operator. We denote its closure by the same symbol. We set

$$a(\Phi) := (a^*(\Phi))^*.$$

$a(\Phi)$ is called the operator of annihilation of Φ. For $w_1, \dots, w_m \in \mathcal{Z}$ we have

$$a^*(w_1 \otimes_{\rm s/a} \cdots \otimes_{\rm s/a} w_m) = a^*(w_1)\cdots a^*(w_m),$$

$$a(w_1 \otimes_{\rm s/a} \cdots \otimes_{\rm s/a} w_m) = a(w_m)\cdots a(w_1).$$

Recall from Subsect. 2.5 that we can identify the space $B^2(\overline{\mathcal{Z}}, \mathcal{Z})$ with $\otimes^2\mathcal{Z}$. Hence, we have an identification of $B^2_{\rm s/a}(\overline{\mathcal{Z}}, \mathcal{Z})$ with $\Gamma^2_{\rm s/a}(\mathcal{Z})$.

Thus if $c \in B^2_{\rm s/a}(\overline{\mathcal{Z}}, \mathcal{Z})$, then by interpreting c as an element of $\Gamma^2_{\rm s/a}(\mathcal{Z})$, we can use the notation $a^*(c)$ / $a(c)$ for the corresponding two-particle creation/annihilation operators.

6 Representations of the CCR in Fock spaces

6.1 Field operators

Let $\mathcal{Z}$ be a (complex) Hilbert space. Define the real vector space

$$\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) := \{(z, \overline{z}) \ : \ z \in \mathcal{Z}\}. \tag{39}$$

Clearly, $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ is a real subspace of $\mathcal{Z} \oplus \overline{\mathcal{Z}}$. For shortness, we will usually write $\mathcal{Y}$ for $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. In this section we will treat $\mathcal{Y}$ as a symplectic space equipped with the symplectic form

$$(z, \overline{z})\omega(w, \overline{w}) = 2\mathrm{Im}(z|w).$$

Consider the creation/annihilation operators $a^*(z)$ and $a(z)$ acting on the bosonic Fock space $\Gamma_{\rm s}(\mathcal{Z})$. For $y = (w, \overline{w}) \in \mathcal{Y}$ we define

$$\phi(y) := a^*(w) + a(w).$$

Note that $\phi(y)$ is essentially self-adjoint on $\Gamma^{\rm fin}_{\rm s}(\mathcal{Z})$. We use the same symbol $\phi(y)$ for its self-adjoint extension.

We have the following commutation relations

$$[\phi(y_1), \phi(y_2)] = \mathrm{i} y_1\omega y_2, \quad y_1, y_2 \in \mathcal{Y},$$

as an identity on $\Gamma^{\rm fin}_{\rm s}(\mathcal{Z})$.

It is well known that in every bosonic Fock space we have a natural representation of the CCR:

Theorem 21. *The map*

$$\mathcal{Y} \ni y \mapsto W(y) := \mathrm{e}^{\mathrm{i}\phi(y)} \in U(\Gamma_{\mathrm{s}}(\mathcal{Z})) \tag{40}$$

is a regular irreducible representation of the CCR.

(40) is called *the Fock representation of the CCR.*

One often identifies the spaces $\mathcal{Y}$ and $\mathcal{Z}$ through

$$\mathcal{Z} \ni z \mapsto \frac{1}{\sqrt{2}}(z, \overline{z}) \in \mathcal{Y}. \tag{41}$$

With this identification, one introduces the field operators for $w \in \mathcal{Z}$ as

$$\phi(w) := \frac{1}{\sqrt{2}} \left(a^*(w) + a(w) \right).$$

The converse identities are

$$a^*(w) = \tfrac{1}{\sqrt{2}} \left(\phi(w) - \mathrm{i}\phi(\mathrm{i}w) \right),$$
$$a(w) = \tfrac{1}{\sqrt{2}} \left(\phi(w) + \mathrm{i}\phi(\mathrm{i}w) \right).$$

Note that in the fermionic case a different identification seems more convenient (see (54)). In this section we will avoid to identify $\mathcal{Z}$ with $\mathcal{Y}$.

Note that the physical meaning of $\mathcal{Z}$ and $\mathcal{Y}$ is different: $\mathcal{Z}$ is the one-particle Hilbert space of the system, $\mathcal{Y}$ is its classical phase space and $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ can be identified with the complexification of the classical phase space, that is $\mathbb{C}\mathcal{Y}$. For instance, if we are interested in a real scalar quantum field theory, then $\mathcal{Z}$ is the space of positive energy solutions of the Klein-Gordon equation, $\mathcal{Y}$ is the space of real solutions and $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ is the space of complex solutions. See e.g. [24] in this collection of lecture notes, where this point is dicussed in more detail.

6.2 Bosonic Gaussian vectors

Let $c \in B^2_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$. Recall that c can be identified with an element of $\Gamma^2_{\mathrm{s}}(\mathcal{Z})$. Recall from Subsect 5.8 that we defined an unbounded operator $a^*(c)$ on $\Gamma_{\mathrm{s}}(\mathcal{Z})$ such that for $\Psi_n \in \Gamma^n_{\mathrm{s}}(\mathcal{Z})$

$$a^*(c)\Psi_n := \sqrt{(n+2)(n+1)}\, c \otimes_{\mathrm{s}} \Psi_n \in \Gamma^{n+2}_{\mathrm{s}}(\mathcal{Z}). \tag{42}$$

Theorem 22. *Assume that* $\|c\| < 1$.

1) $\mathrm{e}^{\frac{1}{2}a^*(c)}$ *is a closable operator on* $\Gamma^{\mathrm{fin}}_{\mathrm{s}}(\mathcal{Z})$.

2) $\det(1 - cc^*) > 0$*, so that we can define the vector*

$$\Omega_c := (\det(1 - cc^*))^{\frac{1}{4}} \exp(\tfrac{1}{2}a^*(c))\Omega \tag{43}$$

It is the unique vector in $\Gamma_{\mathrm{s}}(\mathcal{Z})$ *satisfying*

$$\|\Omega_c\| = 1, \quad (\Omega_c|\Omega) > 0, \quad (a(z) - a^*(c\overline{z}))\Omega_c = 0, \quad z \in \mathcal{Z}.$$

In the Schrödinger representation the vectors Ω_c are normalized Gaussians with an arbitrary dispersion — hence they are often called *squeezed states.*

6.3 Complex structures compatible with a symplectic form

Before analyzing Bogolubov transformations on a Fock space it is natural to start with a little linear algebra of symplectic vector spaces.

We can treat $\mathcal{Y}$ as a real Hilbert space. In fact, we have a natural scalar product

$$(z,\overline{z})\alpha(w,\overline{w}) := \mathrm{Re}(z|w).$$

This scalar product will have a fundamental importance in the next section, when we will discuss fermions. In this section we need it only to define bounded and trace class operators.

We define $Sp(\mathcal{Y})$ to be the set of all bounded invertible linear maps on $\mathcal{Y}$ preserving ω. (This extends the definition of $Sp(\mathcal{Y})$ from the case of a finite dimensional symplectic space $\mathcal{Y}$ to the present context).

A linear map j is called a *complex structure* (or an *antiinvolution*) iff $j^2 = -1$. We say that it is *compatible with a symplectic form* ω iff $j \in Sp(\mathcal{Y})$ and the symmetric form $y_1 \omega j y_2$, where $y_1, y_2 \in \mathcal{Y}$, is positive definite. (One also says that j is Kähler with respect to ω).

On $\mathcal{Y}$ we introduce the linear map

$$j(z,\overline{z}) := (\mathrm{i}z, -\mathrm{i}\overline{z}).$$

It is easy to see that j is a complex structure compatible with ω.

Note that fixing the complex structure j on the symplectic space $\mathcal{Y}$ compatible with the symplectic form ω is equivalent to identifying $\mathcal{Y}$ with $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ for some complex Hilbert space $\mathcal{Z}$.

Let $r \in B(\mathcal{Y})$. We can extend r to $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ by complex linearity. On $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ we can write r as a 2 by 2 matrix

$$r = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix},$$

where $p \in B(\mathcal{Z},\mathcal{Z})$, $q \in B(\overline{\mathcal{Z}},\mathcal{Z})$. Now $r \in Sp(\mathcal{Y})$ iff

$$p^*p - q^{\#}\overline{q} = 1,\ \ p^{\#}\overline{q} - q^*p = 0,$$

$$pp^* - qq^* = 1,\ pq^{\#} - qp^{\#} = 0.$$

We have

$$pp^* \geq 1, \qquad p^*p \geq 1.$$

Hence p^{-1} exists and $\|p^{-1}\| \leq 1$.

We define the operators $c, d \in B(\overline{\mathcal{Z}},\mathcal{Z})$

$$c := p^{-1}q = q^{\#}(p^{\#})^{-1}, \qquad d := q\overline{p}^{-1} = (p^*)^{-1}q^{\#}.$$

Note that d, c are symmetric (in the sense defined in Sect. 2), $\|d\| \leq 1$, $\|c\| \leq 1$,

$$r = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (p^*)^{-1} & 0 \\ 0 & \overline{p} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \overline{c} & 1 \end{bmatrix}, \tag{44}$$

$$1 - cc^* = (p^*p)^{-1}, \quad 1 - dd^* = (\overline{p}^*\overline{p})^{-1}.$$

The decomposition (44) plays an important role in the description of Bogolubov transformations.

In the following theorem we introduce a certain subgroup of $Sp(\mathcal{Y})$, which will play an important role in Shale's Theorem on the implementabilty of Bogolubov transformations.

Theorem 23. *Let $r \in Sp(\mathcal{Y})$. The following conditions are equivalent:*

$$\begin{array}{ll} 0)\ j - rjr^{-1} \in B^2(\mathcal{Y}), & 1)\ rj - jr \in B^2(\mathcal{Y}), \\ 2)\ \mathrm{Tr} q^*q < \infty, & 3)\ \mathrm{Tr}(pp^* - 1) < \infty, \\ 4)\ d \in B^2(\overline{\mathcal{Z}}, \mathcal{Z}), & 5)\ c \in B^2(\overline{\mathcal{Z}}, \mathcal{Z}). \end{array}$$

Define $Sp_2(\mathcal{Y})$ to be the set of $r \in Sp(\mathcal{Y})$ satisfying the above conditions. Then $Sp_2(\mathcal{Y})$ is a group.

6.4 Bosonic Bogolubov transformations in the Fock representation

Consider now the Fock representation $\mathcal{Y} \ni y \mapsto W(y) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ defined in (40).

The following theorem describes when a symplectic transformation is implementable by a unitary transformation. Part 1) was originally proven in [61]. Proof of 1) and 2) can be found in [14, 60].

Theorem 24 (Shale Theorem).

1) *Let $r \in Sp(\mathcal{Y})$. Then the following conditions are equivalent:*
   a) There exists $U \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ such that

$$UW(y)U^* = W(ry), \quad y \in \mathcal{Y}. \tag{45}$$

   b) $r \in Sp_2(\mathcal{Y})$.

2) *If the above conditions are satisfied, then U is defined uniquely up to a phase factor. Moreover, if we define*

$$U_r^j = |\det pp^*|^{-\frac{1}{4}} \mathrm{e}^{-\frac{1}{2}a^*(d)} \Gamma((p^*)^{-1}) \mathrm{e}^{\frac{1}{2}a(c)}, \tag{46}$$

   then U_r^j is the unique unitary operator satisfying (45) and

$$(\Omega | U_r^j \Omega) > 0. \tag{47}$$

3) *If $\mathcal{U}_r = \{\lambda U_r^j \ : \ \lambda \in \mathbb{C},\ |\lambda| = 1\}$, then*

$$Sp_2(\mathcal{Y}) \ni r \mapsto \mathcal{U}_r \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))/U(1)$$

   is a homomorphism of groups.

6.5 Metaplectic group in the Fock representation

$r \mapsto U_r^j$ is not a representation of $Sp(\mathcal{Y})$, it is only a projective representation. By taking a certain subgroup of $Sp_2(\mathcal{Y})$ we can obtain a representation analogous to the metaplectic representation described in Theorem 7.

Define $Sp_1(\mathcal{Y}) := \{r \in Sp(\mathcal{Y}) \ : \ r - 1 \in B^1(\mathcal{Y})\}$. (Recall that $B^1(\mathcal{Y})$ are trace class operators).

Theorem 25. 1) *$Sp_1(\mathcal{Y})$ is a subgroup of $Sp_2(\mathcal{Y})$.*
2) *$r \in Sp_1(\mathcal{Y})$ iff $p - 1 \in B^1(\mathcal{Z})$.*

For $r \in Sp_1(\mathcal{Y})$, define

$$\pm U_r = \pm(\det p^*)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}a^*(d)} \Gamma((p^*)^{-1}) \mathrm{e}^{\frac{1}{2}a(c)}. \tag{48}$$

(We take both signes of the square root, thus $\pm U_r$ denotes a pair of operators differing by a sign).

Theorem 26. 1) *$\pm U_r \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))/\{1 - 1\}$;*
2) *$U_r W(y) U_r^* = W(ry)$.*
3) *The following map is a group homomorphism.*

$$Sp_1(\mathcal{Y}) \ni r \mapsto \pm U_r \in U(\mathcal{H})/\{1, -1\} \tag{49}$$

Clearly, the operators $\pm U_r$ differ by a phase factor from U_r^j from Theorem 24.

6.6 Positive symplectic transformations

Special role is played by positive symplectic transformations. It is easy to show that $r \in Sp_2(\mathcal{Y})$ is a positive self-adjoint operator on $\mathcal{Y}$ iff it is of the form

$$r = \begin{bmatrix} (1 - cc^*)^{-1/2} & (1 - cc^*)^{-1/2} c \\ (1 - c^*c)^{-1/2} c^* & (1 - c^*c)^{-1/2} \end{bmatrix}, \tag{50}$$

for some $c \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$.

The following theorem describes Bogolubov transformations associated with positive symplectic transformations.

Theorem 27. 1) *The formula*

$$R_c := \det(1 - cc^*)^{\frac{1}{4}} \exp(-\tfrac{1}{2} a^*(c)) \Gamma\left((1 - cc^*)^{\frac{1}{2}}\right) \exp(\tfrac{1}{2} a(c)) \tag{51}$$

defines a unitary operator on $\Gamma_{\mathrm{s}}(\mathcal{Z})$.
2) *If Ω_c is defined in (43), then $\Omega = R_c \Omega_c$,*

3)

$$R_c a^*(z) R_c^* = a^*\left((1-cc^*)^{-1/2}z\right) + a\left((1-cc^*)^{-1/2}c\overline{z}\right)$$

$$R_c a(z) R_c^* = a^*\left((1-cc^*)^{-1/2}c\overline{z}\right) + a\left((1-cc^*)^{-1/2}z\right).$$

4) *If r is related to c by (50), then R_c coincides with U_r^{j} defined in (46).*
5) *R_c coincides with U_r defined in (48), where we take the plus sign and the positive square root.*

7 Representations of the CAR in Fock spaces

7.1 Field operators

Let $\mathcal{Z}$ be a Hilbert space. As in the previous section, let $\mathcal{Y} := \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. This time, however, we treat it as a real Hilbert space equipped with the scalar product

$$(z, \overline{z})\alpha(w, \overline{w}) = \mathrm{Re}(z|w).$$

For $w \in \mathcal{Z}$, consider the creation/annihilation operators $a^*(w)$ and $a(w)$ acting on the fermionic Fock space $\Gamma_{\mathrm{a}}(\mathcal{Z})$. For $y = (w, \overline{w}) \in \mathcal{Y}$ we define

$$\phi(y) := a^*(w) + a(w).$$

Note that $\phi(y)$ are bounded and self-adjoint for any $y \in \mathcal{Y}$. Besides,

$$[\phi(y_1), \phi(y_2)]_+ = 2y_1\alpha y_2, \quad y_1, y_2 \in \mathcal{Y}.$$

Thus we have

Theorem 28.

$$\mathcal{Y} \ni y \mapsto \phi(y) \in B_{\mathrm{h}}(\Gamma_{\mathrm{a}}(\mathcal{Z})) \tag{52}$$

is an irreducible representation of the CAR over the space $(\mathcal{Y}, \alpha)$.

(52) is called the *Fock representation of the CAR.*

Let $w_1, \dots, w_m$ be an orthonormal basis of the complex Hilberts space $\mathcal{Z}$. Then

$$(w_1, \overline{w}_1), (-\mathrm{i}w_1, \mathrm{i}\overline{w}_1), \dots (w_m, \overline{w}_m), (-\mathrm{i}w_m, \mathrm{i}\overline{w}_m) \tag{53}$$

is an orthonormal basis of the real Hilbert space $\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. It is easy to see that the orientation of (53) does not depend on the choice of $w_1, \dots, w_m$.

The operator Q defined as in Theorem 9 for this orientation equals the parity operator $I = \Gamma(-1) = (-1)^N$. In fact, using Theorem 9 4), we can compute

$$\begin{aligned} Q &= \mathrm{i}^m \prod_{j=1}^{m} \phi(w_j, \overline{w}_j)\phi(-\mathrm{i}w_j, \mathrm{i}\overline{w}_j) \\ &= \prod_{j=1}^{m} \left(-a^*(w_j)a(w_j) + a(w_j)a^*(w_j)\right) = \Gamma(-1). \end{aligned}$$

In the fermionic case, one often identifies the spaces $\mathcal{Y}$ and $\mathcal{Z}$ through

$$\mathcal{Z} \ni w \mapsto (w, \overline{w}) \in \mathcal{Y}. \tag{54}$$

With this identification, one introduces the field operators for $w \in \mathcal{Z}$ as

$$\phi(w) := a^*(w) + a(w).$$

The converse identities are

$$\begin{aligned} a^*(w) &= \tfrac{1}{2}\left(\phi(w) - \mathrm{i}\phi(\mathrm{i}w)\right), \\ a(w) &= \tfrac{1}{2}\left(\phi(w) + \mathrm{i}\phi(\mathrm{i}w)\right). \end{aligned}$$

Using these identifications, we have for $z, w \in \mathcal{Z}$ the identities

$$\begin{aligned} &[\phi(z), \phi(w)]_+ = 2\mathrm{Re}(w|z), \\ &\Lambda\phi(w)\Lambda = -\mathrm{i}\phi(\mathrm{i}w)I = \mathrm{i}I\phi(\mathrm{i}w), \\ &[\Lambda\phi(z)\Lambda, \phi(w)] = 2\mathrm{Im}(w|z)I, \end{aligned}$$

where I is the parity operator and Λ was introduced in (37).

Note that the identification (54) is different from the one used in the bosonic case (41). In this section we will avoid identifying $\mathcal{Z}$ with $\mathcal{Y}$.

7.2 Fermionic Gaussian vectors

Let $c \in \Gamma_{\mathrm{a}}^2(\mathcal{Z})$. Note that it can be identified with an element of $B_{\mathrm{a}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$. cc^* is trace class, so $\det(1 + cc^*)$ is well defined.

Theorem 29. *Define a vector in $\Gamma_{\mathrm{a}}(\mathcal{Z})$ by*

$$\Omega_c := \left(\det(1 + cc^*)\right)^{-\frac{1}{4}} \exp\left(\tfrac{1}{2}a^*(c)\right)\Omega. \tag{55}$$

It is the unique vector satisfying

$$\|\Omega_c\| = 1, \quad (\Omega_c|\Omega) > 0, \quad (a(z) + a^*(c\overline{z}))\Omega_c = 0, \quad z \in \mathcal{Z}.$$

Vectors of the form Ω_c are often used in the many body quantum theory. In particular, they appear as convenient variational states in theory of superconductivity that goes back to the work of Bardeen-Cooper-Schrieffer, see e.g. [34].

7.3 Complex structures compatible with a scalar product

Similarly as for bosons, it is convenient to study some abstract properties of orthogonal transformations on a real Hilbert space as a preparation for the analysis of fermionic Bogolubov transformations.

Let $O(\mathcal{Y})$ denote the group of orthogonal transformations on $\mathcal{Y}$.

We say that a complex structure j is *compatible with the scalar product* α (or is Kähler with respect to α) if $j \in O(\mathcal{Y})$.

Recall that on $\mathcal{Y}$ we have a distinguished complex structure

$$j(z, \overline{z}) := (\mathrm{i}z, -\mathrm{i}\overline{z}).$$

It is easy to see that j is compatible with α.

Note that fixing the complex structure j on a real Hilbert space $\mathcal{Y}$ compatible with the scalar product α is equivalent with identifying $\mathcal{Y}$ with $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ for some complex Hilbert space $\mathcal{Z}$.

Let $r \in B(\mathcal{Y})$. Recall that we can extend r to $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ by complex linearity and write it as

$$r = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix},$$

where $p \in B(\mathcal{Z}, \mathcal{Z})$, $q \in B(\overline{\mathcal{Z}}, \mathcal{Z})$. Now $r \in O(\mathcal{Y})$ iff

$$p^* p + q^{\#} \overline{q} = 1, \;\; p^{\#} \overline{q} + q^* p = 0,$$

$$pp^* + qq^* = 1, \;\; pq^{\#} + qp^{\#} = 0.$$

It is convenient to distinguish a certain class of orthogonal transformations given by the following theorem:

Theorem 30. *Let $r \in O(\mathcal{Y})$. Then the following conditions are equivalent:*
1) $\mathrm{Ker}(rj + jr) = \{0\}$*;*
2) $\mathrm{Ker}(r^* j + jr^*) = \{0\}$*;*
3) $\mathrm{Ker}\, p = \{0\}$*;*
4) $\mathrm{Ker}\, p^* = \{0\}$.

If the conditions of Theorem 30 are satisfied, then we say that r is *j-nondegenerate*. Let us assume that this is the case. Then p^{-1} and p^{*-1} are densely defined operators. Set

$$d = q\overline{p}^{-1} = -(p^*)^{-1} q^{\#}, \quad c = p^{-1} q = -q^{\#} (p^{\#})^{-1}.$$

and assume that they are bounded. Then $d, c \in B_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$. The following factorization of r plays an important role in the description of fermionic Bogolubov transformations:

$$r = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (p^*)^{-1} & 0 \\ 0 & \overline{p} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \overline{c} & 1 \end{bmatrix}.$$

We also have

$$1 + cc^* = (p^*p)^{-1}, \quad 1 + d^*d = (\overline{pp}^*)^{-1}.$$

The following group will play an important role in the Shale-Stinespring Theorem on the implementability of fermionic Bogolubov transformations:

Proposition 1. *Let $r \in O(\mathcal{Y})$. The following conditions are equivalent:*
1) $j - rjr^{-1} \in B^2(\mathcal{Y})$,
2) $rj - jr \in B^2(\mathcal{Y})$,
3) $q \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$.

Define $O_2(\mathcal{Y})$ to be the set of $r \in O(\mathcal{Y})$ satisfying the above conditions. Then $O_2(\mathcal{Y})$ is a group.

Note that if r is j-nondegenerate, then it belongs to $O_2(\mathcal{Y})$ iff $c \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$, or equivalently, $d \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$.

7.4 Fermionic Bogolubov transformations in the Fock representation

Consider now the Fock representation of the CAR, $\mathcal{Y} \ni y \mapsto \phi(y) \in B_{\mathrm{h}}(\Gamma_{\mathrm{a}}(\mathcal{Z}))$.

Theorem 31. 1) *Let $r \in O(\mathcal{Y})$. Then the following conditions are equivalent:*
a) There exists $U \in U(\Gamma_{\mathrm{a}}(\mathcal{Z}))$ such that

$$U\phi(y)U^* = \phi(ry), \quad y \in \mathcal{Y}. \tag{56}$$

b) $r \in O_2(\mathcal{Y})$

2) *For $r \in O_2(\mathcal{Y})$, the unitary operator U satisfying (56) is defined uniquely up to a phase factor. Let $\mathcal{U}_r$ denote the class of these operators. Then*

$$O_2(\mathcal{Y}) \ni r \mapsto \mathcal{U}_r \in U(\Gamma_{\mathrm{a}}(\mathcal{Z}))/U(1)$$

is a homomorphism of groups.

3) *Let $r \in O_2(\mathcal{Y})$ be j-nondegenerate. Let p, c, d be defined as in the previous subsection. Set*

$$U_r^j = |\det pp^*|^{\frac{1}{4}} \mathrm{e}^{\frac{1}{2}a^*(d)} \Gamma((p^*)^{-1}) \mathrm{e}^{-\frac{1}{2}a(c)}. \tag{57}$$

Then U_r^j is the unique unitary operator satisfying (56) and

$$(\Omega | U_r^j \Omega) > 0. \tag{58}$$

7.5 Pin group in the Fock representation

Define $O_1(\mathcal{Y}) := \{r \in O(\mathcal{Y}) \ : \ r - 1 \in B^1(\mathcal{Y})\}$.

Theorem 32. 1) *$O_1(\mathcal{Y})$ is a subgroup of $O_2(\mathcal{Y})$.*
2) *$r \in O_1(\mathcal{Y})$ iff $p - 1 \in B^1(\mathcal{Z})$.*

The following theorem describes the Pin representation for an arbitrary number of degrees of freedom:

Theorem 33. *There exists a group homomorphism*

$$O_1(\mathcal{Y}) \ni r \mapsto \pm U_r \in U(\Gamma_{\mathrm{a}}(\mathcal{Z}))/\{1, -1\} \tag{59}$$

satisfying $U_r \phi(y) U_r^* = \phi(ry)$.

In order to give a formula for $\pm U_r$, which is analogous to the bosonic formula (48), we have to restrict ourselves to j-nondegenerate transformations.

Theorem 34. *Suppose that $r \in O_1(\mathcal{Y})$ is j-nondegenerate. Then*

$$\pm U_r = \pm(\det p^*)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}a^*(d)} \Gamma((p^*)^{-1}) \mathrm{e}^{-\frac{1}{2}a(c)}. \tag{60}$$

Similarly as in the bosonic case, it is easy to see that the operators $\pm U_r$ differ by a phase factor from U_r^j from Theorem 31.

7.6 *j*-self-adjoint orthogonal transformations

Special role is played by $r \in O_2(\mathcal{Y})$ satisfying $rj = j^* r$. Such transformations will be called *j-self-adjoint*.

One can easily show that $r \in O_2(\mathcal{Y})$ is j-self-adjoint if

$$r = \begin{bmatrix} (1 + cc^*)^{-1/2} & (1 + cc^*)^{-1/2} c \\ -(1 + c^* c)^{-1/2} c^* & (1 + c^* c)^{-1/2} \end{bmatrix}. \tag{61}$$

for some $c \in B^2_{\mathrm{a}}(\mathcal{Y})$.

Theorem 35. 1) *The formula*

$$R_c := \det(1 + cc^*)^{-\frac{1}{4}} \exp(\tfrac{1}{2} a^*(c)) \Gamma\left((1 + cc^*)^{-\frac{1}{2}}\right) \exp(-\tfrac{1}{2} a(c)) \tag{62}$$

defines a unitary operator on $\Gamma_{\mathrm{a}}(\mathcal{Z})$.
2) *If Ω_c is defined in (55), then $\Omega = R_c \Omega_c$.*

3)

$$R_c a^*(z) R_c^* = a^*\left((1+cc^*)^{-1/2}z\right) + a\left((1+cc^*)^{-1/2}c\overline{z}\right),$$

$$R_c a(z) R_c^* = a^*\left((1+cc^*)^{-1/2}c\overline{z}\right) + a\left((1+cc^*)^{-1/2}z\right).$$

4) *If r and c are related by (61), then the operator R_c coincides with the operator U_r^j defined in (57).*
5) *R_c coincides with U_r defined in (34) with the plus sign and the positive square root.*

8 W^*-algebras

In this section we review some elements of the theory of W^*-algebras needed in our paper. For more details we refer the reader to [29], and also [16, 17, 67, 69, 70].

$\mathfrak{M}$ is a *W^*-algebra* if it is a C^*-algebra, possessing a *predual*. (This means that there exists a Banach space $\mathcal{Y}$ such that $\mathfrak{M}$ is isomorphic as a Banach space to the dual of $\mathcal{Y}$. This Banach space $\mathcal{Y}$ is called a predual of $\mathfrak{M}$).

One can show that a predual of a W^*-algebra is defined uniquely up to an isomorphism. The topology on $\mathfrak{M}$ given by the functionals in the predual (the $*$-weak topology in the terminology of theory of Banach spaces) will be called the *σ-weak topology*. The set σ-weakly continuous linear functionals coincides with the predual of $\mathfrak{M}$.

$\mathbb{R} \ni t \mapsto \tau^t$ is called a *W^*-dynamics* if it is a 1-parameter group with values in $*$-automorphisms of $\mathfrak{M}$ and, for any $A \in \mathfrak{M}$, $t \mapsto \tau^t(A)$ is σ-weakly continuous. The pair $(\mathfrak{M}, \tau)$ is called a *W^*-dynamical system.*

$$\mathfrak{M} \cap \mathfrak{M}' := \{B \in \mathfrak{M} \ : \ AB = BA, \ A \in \mathfrak{M}\}$$

is called the *center of the algebra* $\mathfrak{M}$. A W^*-algebra with a trivial center is called a *factor*.

If $\mathfrak{A}$ is a subset of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then

$$\mathfrak{A}' := \{B \ : \ AB = BA, \ A \in \mathfrak{A}\}$$

is called the *commutant of* $\mathfrak{A}$.

8.1 Standard representations

We say that $\mathcal{H}^+$ is a *self-dual cone* in a Hilbert space $\mathcal{H}$ if

$$\mathcal{H}^+ = \{\Phi \in \mathcal{H} \ : \ (\Phi|\Psi) \geq 0, \ \Psi \in \mathcal{H}^+\}.$$

We say that a quadruple $(\pi, \mathcal{H}, J, \mathcal{H}^+)$ is a *standard representation of a* W^**-algebra* $\mathfrak{M}$ if $\pi : \mathfrak{M} \to B(\mathcal{H})$ is an injective σ-weakly continuous $*$-representation, J is an antiunitary involution on $\mathcal{H}$ and $\mathcal{H}^+$ is a self-dual cone in $\mathcal{H}$ satisfying the following conditions:

1) $J\pi(\mathfrak{M})J = \pi(\mathfrak{M})'$;
2) $J\pi(A)J = \pi(A)^*$ for A in the center of $\mathfrak{M}$;
3) $J\Psi = \Psi$ for $\Psi \in \mathcal{H}^+$;
4) $\pi(A)J\pi(A)\mathcal{H}^+ \subset \mathcal{H}^+$ for $A \in \mathfrak{M}$.

Every W^*-algebra has a unique (up to the unitary equivalence) standard representation, [42] (see also [4, 17, 20, 29, 70]).

The standard representation has several important properties. First, every σ-weakly continuous state ω has a unique vector representative in $\mathcal{H}^+$ (in other words, there is a unique normalized vector $\Omega \in \mathcal{H}^+$ such that $\omega(A) = (\Omega|\pi(A)\Omega)$). Secondly, for every $*$-automorphism τ of $\mathfrak{M}$ there exists a unique unitary operator $U \in B(\mathcal{H})$ such that

$$\pi(\tau(A)) = U\pi(A)U^*, \quad U\mathcal{H}^+ \subset \mathcal{H}^+.$$

Finally, for every W^*-dynamics $\mathbb{R} \ni t \mapsto \tau^t$ on $\mathfrak{M}$ there is a unique self-adjoint operator L on $\mathcal{H}$ such that

$$\pi(\tau^t(A)) = \mathrm{e}^{\mathrm{i}tL}\pi(A)\mathrm{e}^{-\mathrm{i}tL}, \quad \mathrm{e}^{\mathrm{i}tL}\mathcal{H}^+ = \mathcal{H}^+. \tag{63}$$

The operator L is called the *standard Liouvillean of* $t \mapsto \tau^t$.

Given a standard representation $(\pi, \mathcal{H}, J, \mathcal{H}^+)$ we also have the *right representation* $\pi_{\mathrm{r}} : \overline{\mathfrak{M}} \to B(\mathcal{H})$ given by $\pi_{\mathrm{r}}(\overline{A}) := J\pi(A)J$. Note that the image of π_{r} is $\pi(\mathfrak{M})'$. We will often write π_{l} for π and call it the *left representation.*

8.2 Tomita-Takesaki theory

Let $\pi : \mathfrak{M} \to B(\mathcal{H})$ be an injective σ-weakly continuous $*$-representation and Ω a cyclic and separating vector for $\pi(\mathfrak{M})$. One then proves that the formula

$$S\pi(A)\Omega = \pi(A)^*\Omega$$

defines a closable antilinear operator S on $\mathcal{H}$. The*modular operator* Δ and the *modular conjugation* J are defined by the polar decomposition of S:

$$S = J\Delta^{1/2}.$$

If we set

$$\mathcal{H}^+ = \{\pi(A)J\pi(A)\Omega \ : \ A \in \mathfrak{M}\}^{\mathrm{cl}},$$

then $(\pi, \mathcal{H}, J, \mathcal{H}^+)$ is a standard representation of $\mathfrak{M}$. Given $\mathfrak{M}$, π and $\mathcal{H}$, it is the unique standard representation with the property $\Omega \in \mathcal{H}^+$.

8.3 KMS states

Let $(\mathfrak{M},\tau)$ be a W^*-dynamical system. Let β be a positive number (having the physical interpretation of the inverse temperature). A σ-weakly continuous state ω on $\mathfrak{M}$ is called a (τ,β)-*KMS state* (or a β-*KMS state for* τ) iff for all $A,B\in\mathfrak{M}$ there exists a function $F_{A,B}$, analytic inside the strip $\{z : 0 < \mathrm{Im}z < \beta\}$, bounded and continuous on its closure, and satisfying the KMS boundary conditions

$$F_{A,B}(t)=\omega(A\tau^t(B)),\qquad F_{A,B}(t+\mathrm{i}\beta)=\omega(\tau^t(B)A).$$

A KMS-state is τ-invariant. If $\mathfrak{M}$ is a factor, then $(\mathfrak{M},\tau)$ can have at most one β-KMS state.

If $\mathfrak{M}\subset B(\mathcal{H})$ and $\Phi\in\mathcal{H}$, we will say that Φ is a (τ,β)-*KMS vector* iff $(\Phi|\cdot\Phi)$ is a (τ,β)-KMS state.

The acronym KMS stands for Kubo-Martin-Schwinger.

8.4 Type I factors—irreducible representation

The most elementary example of a factor is the so-called type I factor – this means the algebra of all bounded operators on a given Hilbert space. In this and the next two subsections we describe various concepts of theory of W^*-algebras on this example.

The space of σ-weakly continuous functionals on $B(\mathcal{H})$ (the predual of $B(\mathcal{H})$) can be identified with $B^1(\mathcal{H})$ (trace class operators) by the formula

$$\psi(A)=\mathrm{Tr}\gamma A,\qquad \gamma\in B^1(\mathcal{H}),\quad A\in B(\mathcal{H}). \tag{64}$$

In particular, σ-weakly continuous states are determined by positive trace one operators, called density matrices. A state given by a density matrix γ is faithful iff $\mathrm{Ker}\gamma=\{0\}$.

If τ is a $*$-automorphism of $B(\mathcal{H})$, then there exists $W\in U(\mathcal{H})$ such that

$$\tau(A)=WAW^*,\quad A\in B(\mathcal{H}). \tag{65}$$

If $t\mapsto\tau^t$ is a W^*-dynamics, then there exists a self-adjoint operator H on $\mathcal{H}$ such that

$$\tau^t(A)=\mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH},\quad A\in B(\mathcal{H}).$$

See e.g. [16].

A state given by (64) is invariant with respect to the W^*-dynamics (65) iff H commutes with γ.

There exists a (β,τ)–KMS state iff $\mathrm{Tr}\,\mathrm{e}^{-\beta H}<\infty$, and then it has the density matrix $\mathrm{e}^{-\beta H}/\mathrm{Tr}\,\mathrm{e}^{-\beta H}$.

8.5 Type I factor—representation in Hilbert-Schmidt operators

Clearly, the representation of $B(\mathcal{H})$ in $\mathcal{H}$ is not in a standard form. To construct a standard form of $B(\mathcal{H})$, consider the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$, denoted $B^2(\mathcal{H})$, and two injective representations:

$$\begin{aligned}
&B(\mathcal{H}) \ni A \mapsto \pi_{\mathrm{l}}(A) \in B(B^2(\mathcal{H})), \quad \pi_{\mathrm{l}}(A)B := AB, \quad B \in B^2(\mathcal{H});\\
&\overline{B(\mathcal{H})} \ni \overline{A} \mapsto \pi_{\mathrm{r}}(\overline{A}) \in B(B^2(\mathcal{H})), \quad \pi_{\mathrm{r}}(\overline{A})B := BA^*, \quad B \in B^2(\mathcal{H}).
\end{aligned} \tag{66}$$

Set $J_{\mathcal{H}}B := B^*$, $B \in B^2(\mathcal{H})$. Then $J_{\mathcal{H}}\pi_{\mathrm{l}}(A)J_{\mathcal{H}} = \pi_{\mathrm{r}}(\overline{A})$ and

$$(\pi_l, B^2(\mathcal{H}), J_{\mathcal{H}}, B^2_+(\mathcal{H}))$$

is a standard representation of $B(\mathcal{H})$.

If a state on $B(\mathcal{H})$ is given by a density matrix $\gamma \in B^1_+(\mathcal{H})$, then its standard vector representative is $\gamma^{\frac{1}{2}} \in B^2_+(\mathcal{H})$. The standard implementation of the $*$-authomorphism $\tau(A) = WAW^*$ equals $\pi_{\mathrm{l}}(W)\pi_r(\overline{W})$. If the W^*-dynamics $t \mapsto \tau^t$ is given by a self-adjoint operator H, then its standard Liouvillean is $\pi_{\mathrm{l}}(H) - \pi_{\mathrm{r}}(\overline{H})$.

8.6 Type I factors—representation in $\mathcal{H} \otimes \overline{\mathcal{H}}$

An alternative formalism, which can be used to describe a standard form of type I factors, uses the notion of a conjugate Hilbert space.

Recall that $B^2(\mathcal{H})$ has a natural identification with $\mathcal{H} \otimes \overline{\mathcal{H}}$. Under the identification the representations (66) become

$$\begin{aligned}
&B(\mathcal{H}) \ni A \mapsto A \otimes 1_{\overline{\mathcal{H}}} \in B(\mathcal{H} \otimes \overline{\mathcal{H}});\\
&\overline{B(\mathcal{H})} \ni \overline{A} \mapsto 1_{\mathcal{H}} \otimes \overline{A} \in B(\mathcal{H} \otimes \overline{\mathcal{H}}).
\end{aligned} \tag{67}$$

(Abusing the notation, sometimes the above representations will also be denoted by π_{l} and π_{r}).

Note that the standard unitary implementation of the automorphism $\tau(A) = WAW^*$ is then equal to $W \otimes \overline{W}$. The standard Liouvillean for $\tau^t(A) = \mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}$ equals $L = H \otimes 1 - 1 \otimes \overline{H}$. The modular conjugation is $J_{\mathcal{H}}$ defined by

$$J_{\mathcal{H}}\big(\Psi_1 \otimes \overline{\Psi_2}\big) := \Psi_2 \otimes \overline{\Psi_1}. \tag{68}$$

The positive cone is then equal to

$$(\mathcal{H} \otimes \overline{\mathcal{H}})_+ := \mathrm{Conv}\{\Psi \otimes \overline{\Psi} \ : \ \Psi \in \mathcal{H}\}^{\mathrm{cl}},$$

where Conv denotes the convex hull.

8.7 Perturbations of W^*-dynamics and Liouvilleans

The material of this subsection will be needed only in the last section devoted to Pauli-Fierz systems.

Let τ be a W^*-dynamics on a W^*-algebra $\mathfrak{M}$ and let $(\pi, \mathcal{H}, J, \mathcal{H}_+)$ be a standard representation of $\mathfrak{M}$. Let L be the standard Liouvillean of τ.

The following theorem is proven in [29]:

Theorem 36. *Let V be a self-adjoint operator on $\mathcal{H}$ affiliated to $\mathfrak{M}$. (That means that all spectral projections of V belong to $\pi(\mathfrak{M})$). Let $L+V$ be essentially self-adjoint on* $\mathrm{Dom}(L)\cap\mathrm{Dom}(V)$ *and $L_V := L+V-JVJ$ be essentially self-adjoint on* $\mathrm{Dom}(L)\cap\mathrm{Dom}(V)\cap\mathrm{Dom}(JVJ)$. *Set*

$$\tau_V^t(A) := \pi^{-1}\left(\mathrm{e}^{\mathrm{i}t(L+V)}\pi(A)\mathrm{e}^{-\mathrm{i}t(L+V)}\right).$$

Then $t \mapsto \tau_V^t$ is a W^*-dynamics on $\mathfrak{M}$ and L_V is its standard Liouvillean.

9 Quasi-free representations of the CCR

9.1 Bosonic quasi-free vectors

Let $(\mathcal{Y}, \omega)$ be a real vector space with an antisymmetric form. Let

$$\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H}) \tag{69}$$

be a representation of the CCR. We say that $\Psi \in \mathcal{H}$ is a *quasi-free vector* for (69) iff there exists a quadratic form η such that

$$(\Psi|W(y)\Psi) = \exp(-\tfrac{1}{4}y\eta y). \tag{70}$$

Note that η is necessarily positive, that is $y\eta y \geq 0$ for $y \in \mathcal{Y}$.

A representation (69) is called *quasi-free* if there exists a cyclic quasi-free vector in $\mathcal{H}$.

The following fact is easy to see:

Theorem 37. *A quasi-free representation is regular.*

Therefore, in a quasi-free representation we can define the corresponding field operators, denoted $\phi(y)$.

If m is an integer, we say that σ is a *pairing of* $\{1,\dots,2m\}$ if it is a permutation of $\{1,\dots,2m\}$ satisfying

$$\sigma(1) < \sigma(3) < \cdots < \sigma(2m-1), \quad \sigma(2j-1) < \sigma(2j),\ j = 1,\dots,m.$$

$P(2m)$ will denote the set of pairings of $\{1,\dots,2m\}$.

Theorem 38. *Suppose we are given a regular representation of the CCR*

$$\mathcal{Y} \ni y \mapsto \mathrm{e}^{\mathrm{i}\phi(y)} \in U(\mathcal{H}).$$

Let $\Psi \in \mathcal{H}$. Then the following statements are equivalent:

1) *For any $n = 1, 2, \ldots,\ y_1, \ldots y_n \in \mathcal{Y}$, $\Psi \in \mathrm{Dom}\,(\phi(y_1)\cdots\phi(y_n))$, and*

$$\begin{aligned}
(\Psi|\phi(y_1)\cdots\phi(y_{2m-1})\Psi) &= 0, \\
(\Psi|\phi(y_1)\cdots\phi(y_{2m})\Psi) &= \sum_{\sigma\in P(2m)}\prod_{j=1}^{m}(\Psi|\phi(y_{\sigma(2j-1)})\phi(y_{\sigma(2j)})\Psi).
\end{aligned}$$

2) *Ψ is a quasi-free vector.*

Theorem 39. *Suppose that Ψ is a quasi-free vector with η satisfying (70). Then*

1) *$y_1(\eta + \frac{\mathrm{i}}{2}\omega)y_2 = (\Psi|\phi(y_1)\phi(y_2)\Psi)$;*
2) *$|y_1\omega y_2| \leq 2|y_1\eta y_1|^{1/2}|y_2\eta y_2|^{1/2}, \quad y_1, y_2 \in \mathcal{Y}$.*

Proof. Note that $(\Psi|\phi(y)^2\Psi) = y\eta y$. This implies

$$\frac{1}{2}\Big((\Psi|\phi(y_1)\phi(y_2)\Psi) + (\Psi|\phi(y_2)\phi(y_1)\Psi)\Big) = y_1\eta y_2.$$

From the canonical commutation relations we get

$$\frac{1}{2}\Big((\Psi|\phi(y_1)\phi(y_2)\Psi) - (\Psi|\phi(y_2)\phi(y_1)\Psi)\Big) = \frac{\mathrm{i}}{2}y_1\omega y_2.$$

This yields 1).

From

$$\|(\phi(y_1) \pm \mathrm{i}\phi(y_2))\Psi\|^2 \geq 0.$$

we get

$$|y_2\omega y_1| \leq y_1\eta y_1 + y_2\eta y_2. \tag{71}$$

This implies 2). □

9.2 Classical quasi-free representations of the CCR

Let us briefly discuss quasi-free representations for the trivial antisymmetric form. In this case the fields commute and can be interpreted as classical random variables, hence we will call such representations *classical*.

Consider a real vector space $\mathcal{Y}$ equipped with a positive scalar product η. Consider the probabilistic Gaussian measure given by the covariance η. That means, if $\dim\mathcal{Y} = n < \infty$, then it is the measure $\mathrm{d}\mu = (\det\eta)^{1/2}(2\pi)^{-n/2}\mathrm{e}^{-y\eta y/2}\mathrm{d}y$, where $\mathrm{d}y$ denotes the Lebesgue measure on $\mathcal{Y}$. If $\dim\mathcal{Y} = \infty$, see e.g. [65].

Consider the Hilbert space $L^2(\mu)$. Note that a dense subspace of of $L^2(\mu)$ can be treated as functions on $\mathcal{Y}$. For $y \in \mathcal{Y}$, let $\phi(y)$ denote the function $\mathcal{Y} \ni v \mapsto y\eta v \in \mathbb{R}$. $\phi(y)$ can be treated as a self-adjoint operator on $L^2(\mu)$.

We equip $\mathcal{Y}$ with the antisymmetric form $\omega = 0$. Then

$$\mathcal{Y} \ni y \mapsto \mathrm{e}^{\mathrm{i}\phi(y)} \in U(L^2(\mu)) \tag{72}$$

is a representation of the CCR.

Let $\Psi \in L^2(\mu)$ be the constant function equal to 1. Then Ψ is a cyclic quasi-free vector for (72).

In the remaining part of this section we will discuss quasi-free representations that are fully "quantum" – whose CCR are given by a non-degenerate antisymmetric form ω.

9.3 Araki-Woods representation of the CCR

In this subsection we describe the *Araki-Woods representations of the CCR* and the corresponding W^*-algebras. These representations were introduced in [8]. They are examples of quasi-free representations. In our presentation we follow [27].

Let $\mathcal{Z}$ be a Hilbert space and consider the Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. We will identify the symplectic space $\mathrm{Re}\big((\mathcal{Z} \oplus \overline{\mathcal{Z}}) \oplus \overline{(\mathcal{Z} \oplus \overline{\mathcal{Z}})}\big)$ with $\mathcal{Z} \oplus \overline{\mathcal{Z}}$, as in (41). Therefore, for $(z_1, \overline{z}_2) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}$, the operator

$$\phi(z_1, \overline{z}_2) := \frac{1}{\sqrt{2}}\big(a^*(z_1, \overline{z}_2) + a(z_1, \overline{z}_2)\big)$$

is the corresponding field operator and $W(z_1, \overline{z}_2) = \mathrm{e}^{\mathrm{i}\phi(z_1, \overline{z}_2)}$ is the corresponding Weyl operator.

We will parametrize the Araki-Woods representation by a self-adjoint operator γ on $\mathcal{Z}$ satisfying $0 \leq \gamma \leq 1$, $\mathrm{Ker}(\gamma - 1) = \{0\}$. Another important object associated to the Araki-Woods representation is a positive operator ρ on $\mathcal{Z}$ called the "1-particle density". It is related to γ by

$$\gamma := \rho(1+\rho)^{-1}, \quad \rho = \gamma(1-\gamma)^{-1}. \tag{73}$$

(Note that in [27] we used ρ to parametrize Araki-Woods representations).

For $z \in \mathrm{Dom}(\rho^{\frac{1}{2}})$ we define two unitary operators on $\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ as:

$$\begin{aligned} W^{\mathrm{AW}}_{\gamma,\mathrm{l}}(z) &:= W\big((\rho+1)^{\frac{1}{2}} z, \overline{\rho}^{\frac{1}{2}} \overline{z}\big), \\ W^{\mathrm{AW}}_{\gamma,\mathrm{r}}(\overline{z}) &:= W\big(\rho^{\frac{1}{2}} z, (\overline{\rho}+1)^{\frac{1}{2}} \overline{z}\big). \end{aligned}$$

We denote by $\mathfrak{M}^{\mathrm{AW}}_{\gamma,\mathrm{l}}$ and $\mathfrak{M}^{\mathrm{AW}}_{\gamma,\mathrm{r}}$ the von Neumann algebras generated by $\{W^{\mathrm{AW}}_{\gamma,\mathrm{l}}(z) : z \in \mathrm{Dom}(\rho^{\frac{1}{2}})\}$ and $\{W^{\mathrm{AW}}_{\gamma,\mathrm{r}}(\overline{z}) : z \in \mathrm{Dom}(\rho^{\frac{1}{2}})\}$ respectively. We

will be call them respectively the *left* and the *right Araki-Woods algebra*. We drop the superscript AW until the end of the section.

The operators τ and ϵ, defined by

$$\mathcal{Z} \oplus \overline{\mathcal{Z}} \ni (z_1, \overline{z}_2) \mapsto \tau(z_1, \overline{z}_2) := (\overline{z}_2, z_1) \in \overline{\mathcal{Z}} \oplus \mathcal{Z}, \tag{74}$$

$$\mathcal{Z} \oplus \overline{\mathcal{Z}} \ni (z_1, \overline{z}_2) \mapsto \epsilon(z_1, \overline{z}_2) := (z_2, \overline{z}_1) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}, \tag{75}$$

will be useful. Note that τ is linear, ϵ antilinear, and

$$\epsilon(z_1, \overline{z}_2) = \overline{\tau(z_1, \overline{z}_2)}. \tag{76}$$

In the following theorem we will describe some basic properties of the Araki-Woods algebras.

Theorem 40. 1) $\mathcal{Z} \supset \mathrm{Dom}(\rho^{\frac{1}{2}}) \ni z \mapsto W_{\gamma,\mathrm{l}}(z) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}))$ *is a regular representation of the CCR. In particular,*

$$W_{\gamma,\mathrm{l}}(z_1) W_{\gamma,\mathrm{l}}(z_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(z_1|z_2)} W_{\gamma,\mathrm{l}}(z_1 + z_2).$$

It will be called the left Araki-Woods representation of the CCR associated to the pair $(\mathcal{Z}, \gamma)$. *The corresponding field, creation and annihilation operators are affiliated to* $\mathfrak{M}_{\gamma,\mathrm{l}}$ *and are given by*

$$\phi_{\gamma,\mathrm{l}}(z) = \phi\Big((\rho+1)^{\frac{1}{2}} z, \overline{\rho}^{\frac{1}{2}} \overline{z}\Big),$$

$$a^*_{\gamma,\mathrm{l}}(z) = a^*\Big((\rho+1)^{\frac{1}{2}} z, 0\Big) + a\Big(0, \overline{\rho}^{\frac{1}{2}} \overline{z}\Big),$$

$$a_{\gamma,\mathrm{l}}(z) = a\Big((\rho+1)^{\frac{1}{2}}\ z, 0\Big) + a^*\Big(0, \overline{\rho}^{\frac{1}{2}} \overline{z}\Big).$$

2) $\overline{\mathcal{Z}} \supset \mathrm{Dom}(\overline{\rho}^{\frac{1}{2}}) \ni \overline{z} \mapsto W_{\gamma,\mathrm{r}}(\overline{z}) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}))$ *is a regular representation of the CCR. In particular*

$$W_{\gamma,\mathrm{r}}(\overline{z}_1) W_{\gamma,\mathrm{r}}(\overline{z}_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(\overline{z}_1|\overline{z}_2)} W_{\gamma,\mathrm{r}}(\overline{z}_1 + \overline{z}_2) = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(z_1|z_2)} W_{\gamma,\mathrm{r}}(\overline{z}_1 + \overline{z}_2).$$

It will be called the right Araki-Woods representation of the CCR associated to the pair $(\mathcal{Z}, \gamma)$. *The corresponding field, creation and annihilation operators are affiliated to* $\mathfrak{M}_{\gamma,\mathrm{r}}$ *and are given by*

$$\phi_{\gamma,\mathrm{r}}(\overline{z}) = \phi\Big(\rho^{\frac{1}{2}} z, (\overline{\rho}+1)^{\frac{1}{2}} \overline{z}\Big),$$

$$a^*_{\gamma,\mathrm{r}}(\overline{z}) = a\Big(\rho^{\frac{1}{2}} z, 0\Big) + a^*\Big(0, (\overline{\rho}+1)^{\frac{1}{2}} \overline{z}\Big),$$

$$a_{\gamma,\mathrm{r}}(\overline{z}) = a^*\Big(\rho^{\frac{1}{2}} z, 0\Big) + a\Big(0, (\overline{\rho}+1)^{\frac{1}{2}} \overline{z}\Big).$$

3) *Set*
$$J_{\mathrm{s}} = \Gamma(\epsilon) \tag{77}$$
Then we have
$$J_{\mathrm{s}} W_{\gamma,\mathrm{l}}(z) J_{\mathrm{s}} = W_{\gamma,\mathrm{r}}(\overline{z}),$$
$$J_{\mathrm{s}} \phi_{\gamma,\mathrm{l}}(z) J_{\mathrm{s}} = \phi_{\gamma,\mathrm{r}}(\overline{z}),$$
$$J_{\mathrm{s}} a^*_{\gamma,\mathrm{l}}(z) J_{\mathrm{s}} = a^*_{\gamma,\mathrm{r}}(\overline{z}),$$
$$J_{\mathrm{s}} a_{\gamma,\mathrm{l}}(z) J_{\mathrm{s}} = a_{\gamma,\mathrm{r}}(\overline{z}).$$
4) *The vacuum Ω is a bosonic quasi-free vector for $W_{\gamma,\mathrm{l}}$, its expectation value for the Weyl operators (the "generating function") is equal to*
$$\big(\Omega|W_{\gamma,\mathrm{l}}(z)\Omega\big) = \exp\Big(-\tfrac{1}{4}(z|z) - \tfrac{1}{2}(z|\rho z)\Big) = \exp\Big(-\tfrac{1}{4}\Big(z|\tfrac{1+\gamma}{1-\gamma} z\Big)\Big)$$
and the "two-point functions" are equal to
$$\begin{aligned}
\big(\Omega|\phi_{\gamma,\mathrm{l}}(z_1)\phi_{\gamma,\mathrm{l}}(z_2)\Omega\big) &= \tfrac{1}{2}(z_1|z_2) + \mathrm{Re}(z_1|\rho z_2),\\
\big(\Omega|a_{\gamma,\mathrm{l}}(z_1)a^*_{\gamma,\mathrm{l}}(z_2)\Omega\big) &= (z_1|(1+\rho)z_2) = (z_1|(1-\gamma)^{-1}z_2),\\
\big(\Omega|a^*_{\gamma,\mathrm{l}}(z_1)a_{\gamma,\mathrm{l}}(z_2)\Omega\big) &= (z_2|\rho z_1) = (z_2|\gamma(1-\gamma)^{-1}z_1),\\
\big(\Omega|a^*_{\gamma,\mathrm{l}}(z_1)a^*_{\gamma,\mathrm{l}}(z_2)\Omega\big) &= 0,\\
\big(\Omega|a_{\gamma,\mathrm{l}}(z_1)a_{\gamma,\mathrm{l}}(z_2)\Omega\big) &= 0.
\end{aligned}$$
5) $\mathfrak{M}_{\gamma,\mathrm{l}}$ *is a factor.*
6) $\mathrm{Ker}\gamma = \{0\}$ *iff Ω is separating for $\mathfrak{M}_{\gamma,\mathrm{l}}$ iff Ω is cyclic for $\mathfrak{M}_{\gamma,\mathrm{l}}$. If this is the case, then the modular conjugation for Ω is given by (77) and the modular operator for Ω is given by*
$$\Delta = \Gamma\left(\gamma \oplus \overline{\gamma}^{-1}\right). \tag{78}$$
7) *We have*
$$\mathfrak{M}'_{\gamma,\mathrm{l}} = \mathfrak{M}_{\gamma,\mathrm{r}}. \tag{79}$$
8) *Define*
$$\Gamma^+_{\mathrm{s},\gamma}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) := \{A J_{\mathrm{s}} A \Omega \ : \ A \in \mathfrak{M}_{\gamma,\mathrm{l}}\}^{\mathrm{cl}}. \tag{80}$$
Then $(\mathfrak{M}_{\gamma,\mathrm{l}}, \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}), J_{\mathrm{s}}, \Gamma^+_{\mathrm{s},\gamma}(\mathcal{Z} \oplus \overline{\mathcal{Z}}))$ is a W^*-algebra in the standard form.
9) *If γ has some continuous spectrum, then $\mathfrak{M}_{\gamma,\mathrm{l}}$ is a factor of type* III_1 *[69].*
10) *If $\gamma = 0$, then $\mathfrak{M}_{\gamma,\mathrm{l}}$ is a factor of type* I.

11) *Let h be a self-adjoint operator on $\mathcal{Z}$ commuting with γ and*

$$\tau^t(W_{\gamma,\mathrm{l}}(z)) := W_{\gamma,\mathrm{l}}(\mathrm{e}^{\mathrm{i}th}z).$$

Then $t \mapsto \tau^t$ extends to a W^*-dynamics on $\mathfrak{M}_{\gamma,\mathrm{l}}$ and

$$L = \mathrm{d}\Gamma(h \oplus (-\overline{h}))$$

is its standard Liouvillean.

12) *Ω is a (τ,β)-KMS vector iff $\gamma = \mathrm{e}^{-\beta h}$.*

Proof. 1)–4) follow by straightforward computations.

Let us prove 5). We have

$$W_{\gamma,\mathrm{l}}(z_1)W_{\gamma,\mathrm{r}}(\overline{z}_2) = W_{\gamma,\mathrm{r}}(\overline{z}_2)W_{\gamma,\mathrm{l}}(z_1), \quad z_1, z_2 \in \mathrm{Dom}\rho^{\frac{1}{2}}.$$

Consequently, $\mathfrak{M}_{\gamma,\mathrm{l}}$ and $\mathfrak{M}_{\gamma,\mathrm{r}}$ commute with one another.

Now

$$\begin{aligned}
&\big(\mathfrak{M}_{\gamma,\mathrm{l}} \cup \mathfrak{M}'_{\gamma,\mathrm{l}}\big)' \subset \big(\mathfrak{M}_{\gamma,\mathrm{l}} \cup \mathfrak{M}_{\gamma,\mathrm{r}}\big)' \\
&= \big\{W((\rho+1)^{\frac{1}{2}}z_1 + \rho^{\frac{1}{2}}z_2, \overline{\rho}^{\frac{1}{2}}\overline{z}_1 + (\overline{\rho}+1)^{\frac{1}{2}}\overline{z}_2\big) \ : \ z_1, z_2 \in \mathcal{Z}\big\}' \\
&= \{W(w_1, \overline{w}_2) \ : \ w_1, w_2 \in \mathcal{Z}\}' = \mathbb{C}1,
\end{aligned}$$

because

$$\big\{\big((\rho+1)^{\frac{1}{2}}z_1 + \rho^{\frac{1}{2}}z_2, \overline{\rho}^{\frac{1}{2}}\overline{z}_1 + (\overline{\rho}+1)^{\frac{1}{2}}\overline{z}_2\big) \ : \ z_1, z_2 \in \mathcal{Z}\big\}$$

is dense in $\mathcal{Z} \oplus \overline{\mathcal{Z}}$, and Weyl operators depend strongly continuously on their parameters and act irreducibly on $\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. Therefore,

$$\big(\mathfrak{M}_{\gamma,\mathrm{l}} \cup \mathfrak{M}'_{\gamma,\mathrm{l}}\big)' = \mathbb{C}1,$$

which means that $\mathfrak{M}_{\gamma,\mathrm{l}}$ is a factor and proves 5).

Let us prove the $\Rightarrow$ part of 6). Assume first that $\mathrm{Ker}\gamma = \{0\}$. Set $\tau^t(A) := \Gamma(\gamma, \overline{\gamma}^{-1})^{\mathrm{i}t} A \Gamma(\gamma, \overline{\gamma}^{-1})^{-\mathrm{i}t}$. We first check that τ^t preseves $\mathfrak{M}_{\gamma,\mathrm{l}}$. Therefore, it is a W^*-dynamics on $\mathfrak{M}_{\gamma,\mathrm{l}}$.

Next we check that $(\Omega| \cdot \Omega)$ satisfies the $(\tau, -1)$-KMS condition. This is straightforward for the Weyl operators $W_{\gamma,\mathrm{l}}(z)$. Therefore, it holds for the $*-$algebra $\mathfrak{M}_{\gamma,\mathrm{l},0}$ of finite linear combinations of $W_{\gamma,\mathrm{l}}(z)$. By the Kaplansky Theorem, the unit ball of $\mathfrak{M}_{\gamma,\mathrm{l},0}$ is σ-weakly dense in the unit ball of $\mathfrak{M}_{\gamma,\mathrm{l}}$. Using this we extend the KMS condition to $\mathfrak{M}_{\gamma,\mathrm{l}}$.

A KMS state on a factor is always faithful. By 5), $\mathfrak{M}_{\gamma,\mathrm{l}}$ is a factor. Hence Ω is separating.

Let $\mathcal{H}$ be the closure of $\mathfrak{M}_{\gamma,\mathrm{l}}\Omega$. $\mathcal{H}$ is an invariant subspace for $\mathfrak{M}_{\gamma,\mathrm{l}}$, moreover Ω is cyclic and separating for $\mathfrak{M}_{\gamma,\mathrm{l}}$ on $\mathcal{H}$. Let us compute the operators S, Δ and J of the modular theory for Ω on $\mathcal{H}$.

Clearly $\mathcal{H}$ is spanned by vectors of the form $\Psi_z := \mathrm{e}^{\mathrm{i}a^*\left((1+\rho)^{\frac{1}{2}}z,\overline{\rho}^{\frac{1}{2}}\overline{z}\right)}\Omega$. Let us compute:

$$\begin{aligned}\Gamma(\gamma,\overline{\gamma}^{-1})^{\frac{1}{2}}\Psi_z &= \mathrm{e}^{\mathrm{i}a^*\left(\rho^{\frac{1}{2}}z,(1+\overline{\rho})^{\frac{1}{2}}\overline{z}\right)}\Omega,\\ S\Psi_z &= \mathrm{e}^{-\mathrm{i}a^*\left((1+\rho)^{\frac{1}{2}}z,\overline{\rho}^{\frac{1}{2}}\overline{z}\right)}\Omega\\ &= J_{\mathrm{s}}\Gamma(\gamma,\overline{\gamma}^{-1})^{\frac{1}{2}}\Psi_z.\end{aligned}$$

We have

$$\Gamma(\gamma,\overline{\gamma}^{-1})^{\mathrm{i}t}\Psi_z = \Psi_{\gamma^{\mathrm{i}t}z}.$$

Hence $\Gamma(\gamma,\overline{\gamma}^{-1})$ preserves $\mathcal{H}$ and vectors Ψ_z form an essential domain for its restriction to $\mathcal{H}$. Besides, $\Gamma(\gamma,\overline{\gamma}^{-1})\mathcal{H}$ is dense in $\mathcal{H}$ and S preserves $\mathcal{H}$ as well. Therefore, J_{s} preserves $\mathcal{H}$. Thus

$$S = J_{\mathrm{s}}\Big|_{\mathcal{H}}\,\Gamma(\gamma,\overline{\gamma}^{-1})\Big|_{\mathcal{H}}$$

is the the polar decomposition of S and defines the modular operator and the modular conjugation. Next we see that

$$W_{\gamma,\mathrm{l}}(z_1)J_{\mathrm{s}}W_{\gamma,\mathrm{l}}(z_2)\Omega = W\big((\rho+1)^{\frac{1}{2}}z_1+\rho^{\frac{1}{2}}z_2,\overline{\rho}^{\frac{1}{2}}\overline{z}_1+(\overline{\rho}+1)^{\frac{1}{2}}\overline{z}_2\big)\Omega.$$

Therefore, $\mathfrak{M}_{\gamma,\mathrm{l}}J_{\mathrm{s}}\mathfrak{M}_{\gamma,\mathrm{l}}\Omega$ is dense in $\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$. But $\mathfrak{M}_{\gamma,\mathrm{l}}J_{\mathrm{s}}\mathfrak{M}_{\gamma,\mathrm{l}}\Omega\subset\mathcal{H}$. Hence $\mathcal{H}=\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ and Ω is cyclic. This proves the $\Rightarrow$ part of 6).

To prove 7), we first assume that $\mathrm{Ker}\gamma=\{0\}$. By 6), we can apply the modular theory, which gives $\mathfrak{M}_{\gamma,\mathrm{l}}' = J_{\mathrm{s}}\mathfrak{M}_{\gamma,\mathrm{l}}J_{\mathrm{s}}$. By 3) we have $J_{\mathrm{s}}\mathfrak{M}_{\gamma,\mathrm{l}}J_{\mathrm{s}} = \mathfrak{M}_{\gamma,\mathrm{r}}$.

For a general γ, we decompose $\mathcal{Z}=\mathcal{Z}_0\oplus\mathcal{Z}_1$, where $\mathcal{Z}_0=\mathrm{Ker}\gamma$ and $\mathcal{Z}_1$ is equipped with a nondegenerate $\gamma_1 := \gamma\big|_{\mathcal{Z}_1}$. We then have $\mathfrak{M}_{\gamma,\mathrm{l}}\simeq B(\Gamma_{\mathrm{s}}(\mathcal{Z}_0))\otimes\mathfrak{M}_{\gamma_1,\mathrm{l}}$ and $\mathfrak{M}_{\gamma,\mathrm{r}}\simeq B(\Gamma_{\mathrm{s}}(\overline{\mathcal{Z}}_0))\otimes\mathfrak{M}_{\gamma_1,\mathrm{r}}$. This implies that $\mathfrak{M}_{\gamma,\mathrm{l}}'=\mathfrak{M}_{\gamma,\mathrm{r}}$ and ends the proof of 7).

From the decomposition $\mathfrak{M}_{\gamma,\mathrm{l}}\simeq B(\Gamma_{\mathrm{s}}(\mathcal{Z}_0))\otimes\mathfrak{M}_{\gamma_1,\mathrm{l}}$ we see that if $\mathrm{Ker}\gamma = \mathcal{Z}_0\neq\{0\}$, then Ω is neither cyclic nor separating. This completes the proof of 6) [32]. □

9.4 Quasi-free representations of the CCR as the Araki-Woods representations

A large class of quasi-free representation is unitarily equivalent to the Araki-Woods representation for some γ.

Theorem 41. *Suppose that we are given a representation of the CCR*

$$\mathcal{Y}_0\ni y\mapsto W(y)\in U(\mathcal{H}), \tag{81}$$

with a cyclic quasi-free vector Ψ satisfying $(\Psi|W(y)\Psi) = \mathrm{e}^{-\frac{1}{4}y\eta y}$. Suppose that the symmetric form η is nondegenerate. Let $\mathcal{Y}$ be the completion of $\mathcal{Y}_0$ to a real Hilbert space with the scalar product given by η. By Theorem 39 2), ω extends to a bounded antisymmetric form on $\mathcal{Y}$, which we denote alsoby ω. Assume that ω is nondegenerate on $\mathcal{Y}$. Then there exists a Hilbert space $\mathcal{Z}$ and a positive operator γ on $\mathcal{Z}$, a linear injection of $\mathcal{Y}_0$ onto a dense subspace of $\mathcal{Z}$ and an isometric operator $U : \mathcal{H} \to \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ such that

$$U\Psi = \Omega,$$

$$UW(y) = W_{\gamma,\mathrm{l}}(y)U, \quad y \in \mathcal{Y}_0.$$

Proof. Without loss of generality we can assume that $\mathcal{Y}_0 = \mathcal{Y}$.

Working in the real Hilbert space $\mathcal{Y}$ equipped with the scalar product η and using Theorem 39 2), we see that ω is a bilinear form bounded by 2. Therefore there exist a bounded antisymmetric operator μ with a trivial kernel, $\|\mu\| \leq 1$, such that

$$y_1\omega y_2 = 2y_1\eta\mu y_2.$$

Consider the polar decomposition

$$\mu = |\mu|j = j|\mu|.$$

Then j is an orthogonal operator satisfying $j^2 = -1$.

Let $\mathcal{Z}$ be the completion of $\mathcal{Y}$ with respect to the scalar product $\eta|\mu|$. Then j maps $\mathcal{Z}$ into itself and is an orthogonal operator for the scalar product $\eta|\mu|$ satisfying $j^2 = -1$. We can treat $\mathcal{Z}$ as a complex space, identifying $-j$ with the imaginary unit. We equip it with the (sesquilinear) scalar product

$$(y_1|y_2) := y_1\eta|\mu|y_2 + \mathrm{i}y_1\eta\mu y_2 = y_1\omega j y_2 + \mathrm{i}y_1\omega y_2.$$

$\rho := |\mu|^{-1} - 1$ defines a positive operator on $\mathcal{Z}$ such that $\mathcal{Y} = \mathrm{Dom}\rho^{\frac{1}{2}}$. Now

$$\begin{aligned}(\Psi|\phi(y_1)\phi(y_2)\Psi) &= y_1\eta y_2 + \tfrac{\mathrm{i}}{2}y_1\omega y_2 \\ &= y_1\eta|\mu|y_2 + \mathrm{i}y_1\eta\mu y_2 + y_1\eta|\mu|(|\mu|^{-1} - 1)y_2 \\ &= (y_1|y_2) + \mathrm{Re}(y_1|\rho y_2) \\ &= (\Omega|\phi_{\gamma,\mathrm{l}}(y_1)\phi_{\gamma,\mathrm{l}}(y_2)\Omega),\end{aligned}$$

for γ as in (73). Therefore,

$$UW(y)\Psi := W_{\gamma,\mathrm{l}}(y)\Omega$$

extends to an isometric map from $\mathcal{H}$ to $\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ that intertwines the representation (81) with the Araki-Woods representation for $(\mathcal{Z}, \gamma)$. □

10 Quasi-free representations of the CAR

10.1 Fermionic quasi-free vectors

Let $(\mathcal{Y},\alpha)$ be a real Hilbert space. Let

$$\mathcal{Y} \ni y \mapsto \phi(y) \in B_{\mathrm{h}}(\mathcal{H}) \tag{82}$$

be a representation of the CAR. We say that $\Psi \in \mathcal{H}$ is a *quasi-free vector* for (82) iff

$$(\Psi|\phi(y_1)\cdots\phi(y_{2m-1})\Psi) = 0,$$

$$(\Psi|\phi(y_1)\cdots\phi(y_{2m})\Psi) = \sum_{\sigma\in P(2m)} \mathrm{sgn}\sigma \prod_{j=1}^{m} (\Psi|\phi(y_{\sigma(2j-1)})\phi(y_{\sigma(2j)})\Psi).$$

We say that (82) is a *quasi-free representation* if there exists a cyclic quasi-free vector Ψ in $\mathcal{H}$.

Define the antisymmetric form ω

$$y_1\omega y_2 := \frac{1}{\mathrm{i}}(\Psi|[\phi(y_1),\phi(y_2)]\Psi). \tag{83}$$

Theorem 42. 1) $(\Psi|\phi(y_1)\phi(y_2)\Psi) = y_1\alpha y_2 + \frac{\mathrm{i}}{2} y_1\omega y_2$;
2) $|y_1\omega y_2| \leq 2|y_1\alpha y_1|^{\frac12}|y_2\alpha y_2|^{\frac12}$.

10.2 Araki-Wyss representation of the CAR

In this subsection we describe *Araki-Wyss representations of the CAR* [9], see also [44]. They are examples of quasi-free representations of the CAR.

Let $\mathcal{Z}$ be a Hilbert space and consider the Fock space $\Gamma_{\mathrm{a}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$. We will identify the real Hilbert space $\mathrm{Re}\big((\mathcal{Z}\oplus\overline{\mathcal{Z}})\oplus\overline{(\mathcal{Z}\oplus\overline{\mathcal{Z}})}\big)$ with $\mathcal{Z}\oplus\overline{\mathcal{Z}}$, as in (54). Therefore, for $(z_1,\overline{z}_2)\in\mathcal{Z}\oplus\overline{\mathcal{Z}}$,

$$\phi(z_1,\overline{z}_2) := a^*(z_1,\overline{z}_2) + a(z_1,\overline{z}_2)$$

are the corresponding field operators.

We will parametrize Araki-Wyss representation by a positive operator γ on $\mathcal{Z}$, possibly with a non-dense domain. We will also use the operator χ, called the "1-particle density", which satisfies $0 \leq \chi \leq 1$. The two operators are related to one another by

$$\gamma := \chi(1-\chi)^{-1}, \quad \chi = \gamma(\gamma+1)^{-1}. \tag{84}$$

For $z\in\mathcal{Z}$ we define the Araki-Wyss field operators on $\Gamma_{\mathrm{a}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ as:

$$\phi^{\mathrm{AW}}_{\gamma,\mathrm{l}}(z) := \phi\big((1-\chi)^{\frac12}z, \overline{\chi}^{\frac12}\overline{z}\big),$$

$$\phi^{\mathrm{AW}}_{\gamma,\mathrm{r}}(\overline{z}) := \Lambda\phi\big(\chi^{\frac12}z, (1-\overline{\chi})^{\frac12}\overline{z}\big)\Lambda = \mathrm{i}I\phi\big(\mathrm{i}\chi^{\frac12}z, \mathrm{i}(1-\overline{\chi})^{\frac12}\overline{z}\big),$$

where recall that $\Lambda = (-1)^{N(N-1)/2}$ and $I = (-1)^N = \Gamma(-1)$. The maps $z \mapsto \phi_{\gamma,\mathrm{l}}(z)$, $\overline{z} \mapsto \phi_{\gamma,\mathrm{r}}(\overline{z})$, are called respectively the left and the right Araki-Wyss representation of the CAR associated to the pair $(\mathcal{Z}, \gamma)$. We denote by $\mathfrak{M}^{\mathrm{AW}}_{\gamma,\mathrm{l}}$ and $\mathfrak{M}^{\mathrm{AW}}_{\gamma,\mathrm{r}}$ the von Neumann algebras generated by $\{\phi_{\gamma,\mathrm{l}}(z) : z \in \mathcal{Z}\}$ and $\{\phi_{\gamma,\mathrm{r}}(\overline{z}) : z \in \mathcal{Z}\}$. They will be called respectively the *left* and the *right Araki-Wyss algebra*.

We drop the superscript AW until the end of the section.

In the following theorem we will describe some basic properties of the Araki-Wyss algebras.

Theorem 43. 1) $\mathcal{Z} \ni z \mapsto \phi_{\gamma,\mathrm{l}}(z) \in B_{\mathrm{h}}(\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}))$ *is a representation of the CAR. In particular,*

$$[\phi_{\gamma,\mathrm{l}}(z_1), \phi_{\gamma,\mathrm{l}}(z_2)]_+ = 2\mathrm{Re}(z_1|z_2).$$

The corresponding creation and annihilation operators belong to $\mathfrak{M}_{\gamma,\mathrm{l}}$ *and are given by*

$$a^*_{\gamma,\mathrm{l}}(z) = a^*\Big((1-\chi)^{\frac{1}{2}} z, 0\Big) + a\Big(0, \overline{\chi}^{\frac{1}{2}} \overline{z}\Big),$$
$$a_{\gamma,\mathrm{l}}(z) = a\Big((1-\chi)^{\frac{1}{2}} z, 0\Big) + a^*\Big(0, \overline{\chi}^{\frac{1}{2}} \overline{z}\Big).$$

2) $\overline{\mathcal{Z}} \ni \overline{z} \mapsto \phi_{\gamma,\mathrm{r}}(\overline{z}) \in B_{\mathrm{h}}(\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}))$ *is a representation of the CAR. In particular*

$$[\phi_{\gamma,\mathrm{r}}(\overline{z}_1), \phi_{\gamma,\mathrm{r}}(\overline{z}_2)]_+ = 2\mathrm{Re}(z_1|z_2).$$

The corresponding creation and annihilation operators belong to $\mathfrak{M}_{\gamma,\mathrm{r}}$ *and are given by*

$$a^*_{\gamma,\mathrm{r}}(z) = \Lambda\Big(a\big(\chi^{\frac{1}{2}} z, 0\big) + a^*\big(0, (1-\overline{\chi})^{\frac{1}{2}} \overline{z}\big)\Big)\Lambda,$$
$$a_{\gamma,\mathrm{r}}(z) = \Lambda\Big(a^*\big(\chi^{\frac{1}{2}} z, 0\big) + a\big(0, (1-\overline{\chi})^{\frac{1}{2}} \overline{z}\big)\Big)\Lambda.$$

3) *Set*

$$J_{\mathrm{a}} := \Lambda\Gamma(\epsilon). \tag{85}$$

We have

$$J_{\mathrm{a}}\phi_{\gamma,\mathrm{l}}(z)J_{\mathrm{a}} = \phi_{\gamma,\mathrm{r}}(\overline{z}),$$

$$J_{\mathrm{a}}a^*_{\gamma,\mathrm{l}}(z)J_{\mathrm{a}} = a^*_{\gamma,\mathrm{r}}(\overline{z}),$$

$$J_{\mathrm{a}}a_{\gamma,\mathrm{l}}(z)J_{\mathrm{a}} = a_{\gamma,\mathrm{r}}(\overline{z}).$$

4) *The vacuum Ω is a fermionic quasi-free vector, the "two-point functions" are equal*

$$(\Omega|\phi_{\gamma,\mathrm{l}}(z_1)\phi_{\gamma,\mathrm{l}}(z_2)\Omega) = (z_1|z_2) - \mathrm{i}2\mathrm{Im}(z_1|\chi z_2),$$

$$\big(\Omega|a_{\gamma,\mathrm{l}}(z_1)a^*_{\gamma,\mathrm{l}}(z_2)\Omega\big) = (z_1|(1-\chi)z_2) = (z_1|(1+\gamma)^{-1}z_2),$$

$$\big(\Omega|a^*_{\gamma,\mathrm{l}}(z_1)a_{\gamma,\mathrm{l}}(z_2)\Omega\big) = (z_2|\chi z_1) = (z_2|\gamma(\gamma+1)^{-1}z_1),$$

$$\big(\Omega|a^*_{\gamma,\mathrm{l}}(z_1)a^*_{\gamma,\mathrm{l}}(z_2)\Omega\big) = 0,$$

$$\big(\Omega|a_{\gamma,\mathrm{l}}(z_1)a_{\gamma,\mathrm{l}}(z_2)\Omega\big) = 0.$$

5) $\mathfrak{M}_{\gamma,\mathrm{l}}$ *is a factor.*
6) $\mathrm{Ker}\gamma = \mathrm{Ker}\gamma^{-1} = \{0\}$ *(equivalently,* $\mathrm{Ker}\chi = \mathrm{Ker}(1-\chi) = \{0\}$*) iff Ω is separating for $\mathfrak{M}_{\gamma,\mathrm{l}}$ iff Ω is cyclic for $\mathfrak{M}_{\gamma,\mathrm{l}}$. If this is the case, then the modular conjugation for Ω is given by (85) and the modular operator for Ω is given by*
$$\Delta = \Gamma\left(\gamma \oplus \overline{\gamma}^{-1}\right). \tag{86}$$
7) *We have*
$$\mathfrak{M}'_{\gamma,\mathrm{l}} = \mathfrak{M}_{\gamma,\mathrm{r}}. \tag{87}$$
8) *Set*
$$\Gamma^+_{\mathrm{a},\gamma}(\mathcal{Z}\oplus\overline{\mathcal{Z}}) := \{AJ_{\mathrm{a}}A\Omega \ : \ A \in \mathfrak{M}_{\gamma,\mathrm{l}}\}^{\mathrm{cl}}. \tag{88}$$
Then $(\mathfrak{M}_{\gamma,\mathrm{l}}, \Gamma_{\mathrm{a}}(\mathcal{Z}\oplus\overline{\mathcal{Z}}), J_{\mathrm{a}}, \Gamma^+_{\mathrm{a},\gamma}(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$ is a W^*-algebra in the standard form.
9) *If γ has some continuous spectrum, then $\mathfrak{M}_{\gamma,\mathrm{l}}$ is a factor of type* III_1.
10) *If $\mathcal{Z}$ is an infinite dimensional Hilbert space and $\gamma = \lambda$ or $\gamma = \lambda^{-1}$ with $\lambda \in]0,1[$, then $\mathfrak{M}_{\gamma,\mathrm{l}}$ is a factor of type* III_λ*, [70].*
11) *If $\mathcal{Z}$ is an infinite dimensional Hilbert space and $\gamma = 1$ (equivalently, $\chi = \frac{1}{2}$), then $\mathfrak{M}_{\gamma,\mathrm{l}}$ is a factor of type* II_1.
12) *If $\gamma = 0$ or $\gamma^{-1} = 0$, (equivalently, $\chi = 0$ or $\chi = 1$), then $\mathfrak{M}_{\gamma,\mathrm{l}}$ is a factor of type* I.
13) *Let h be a self-adjoint operator on $\mathcal{Z}$ commuting with γ and*
$$\tau^t(\phi_{\gamma,\mathrm{l}}(z)) := \phi_{\gamma,\mathrm{l}}(\mathrm{e}^{\mathrm{i}th}z).$$
Then $t \mapsto \tau^t$ extends uniquely to a W^*-dynamics on $\mathfrak{M}_{\gamma,\mathrm{l}}$ and
$$L = \mathrm{d}\Gamma(h \oplus (-\overline{h}))$$
is its standard Liouvillean.
14) *Ω is a (τ,β)-KMS vector iff $\gamma = \mathrm{e}^{-\beta h}$.*

Proof. 1)-4) follow by direct computations.

The proof of 5) will be divided into a number of steps.

Step 1. We have

$$\phi_{\gamma,\mathrm{l}}(z_1)\phi_{\gamma,\mathrm{r}}(\overline{z}_2) = \phi_{\gamma,\mathrm{r}}(\overline{z}_2)\phi_{\gamma,\mathrm{l}}(z_1).$$

Consequently, $\mathfrak{M}_{\gamma,\mathrm{l}}$ and $\mathfrak{M}_{\gamma,\mathrm{r}}$ commute with one another.

Step 2. For simplicity, in this step we assume that $\mathcal{Z}$ is separable and $0 \leq \chi \leq \frac{1}{2}$; the generalization of the proof to the general case is easy. By a well known theorem (see e.g. [69], Ex. II.1.4), for any $\epsilon > 0$, we can find a self-adjoint operator ν such that $\mathrm{Tr}(\chi^{\frac{1}{2}} - \nu^{\frac{1}{2}})^*(\chi^{\frac{1}{2}} - \nu^{\frac{1}{2}}) < \epsilon^2$ and there exists an orthonormal basis $w_1, w_2, \dots$ of eigenvectors of ν. Let $\nu w_i = \nu_i w_i$, $\nu_i \in \mathbb{R}$.

Introduce the operators

$$\begin{aligned}
A_j &:= \phi\left((1-\nu_j)^{\frac{1}{2}} w_j, \overline{\nu}_j^{\frac{1}{2}}\overline{w}_j\right)\phi\left(\mathrm{i}(1-\nu_j)^{\frac{1}{2}} w_j, -\mathrm{i}\overline{\nu}_j^{\frac{1}{2}}\overline{w}_j\right) \\
&\quad\times \Lambda\phi\left(\nu_j^{\frac{1}{2}} w_j, (1-\overline{\nu}_j)^{\frac{1}{2}}\overline{w}_j\right)\phi\left(\mathrm{i}\nu_j^{\frac{1}{2}} w_j, -\mathrm{i}(1-\overline{\nu}_j)^{\frac{1}{2}}\overline{w}_j\right)\Lambda \\
&:= \phi\left((1-\nu_j)^{\frac{1}{2}} w_j, \overline{\nu}_j^{\frac{1}{2}}\overline{w}_j\right)\phi\left(\mathrm{i}(1-\nu_j)^{\frac{1}{2}} w_j, -\mathrm{i}\overline{\nu}_j^{\frac{1}{2}}\overline{w}_j\right) \\
&\quad\times \phi\left(\mathrm{i}\nu_j^{\frac{1}{2}} w_j, \mathrm{i}(1-\overline{\nu}_j)^{\frac{1}{2}}\overline{w}_j\right)\phi\left(-\nu_j^{\frac{1}{2}} w_j, (1-\overline{\nu}_j)^{\frac{1}{2}}\overline{w}_j\right) \\
&= \left(2a^*(w_j,0)a(w_j,0)-1\right)\left(2a^*(0,\overline{w}_j)a(0,\overline{w}_j)-1\right).
\end{aligned}$$

Note that A_j commute with one another and $\prod_{j=1}^{\infty} A_j$ converges in the σ-weak topology to I.

Introduce also

$$\begin{aligned}
B_j &:= \phi_{\gamma,\mathrm{l}}(w_j)\phi_{\gamma,\mathrm{l}}(\mathrm{i}w_j)\phi_{\gamma,\mathrm{r}}(\overline{w}_j)\phi_{\gamma,\mathrm{r}}(\mathrm{i}\overline{w}_j) \\
&= \left(2a^*\left((1-\chi)^{\frac{1}{2}} w_j, \overline{\chi}^{\frac{1}{2}}\overline{w}_j\right) a\left((1-\chi)^{\frac{1}{2}} w_j, \overline{\chi}^{\frac{1}{2}}\overline{w}_j\right) - 1\right) \\
&\quad\times\left(2a^*\left(-\chi^{\frac{1}{2}} w_j, (1-\overline{\chi})^{\frac{1}{2}}\overline{w}_j\right) a\left(-\chi^{\frac{1}{2}} w_j, (1-\overline{\chi})^{\frac{1}{2}}\overline{w}_j\right) - 1\right)
\end{aligned}$$

Note that B_j belongs to the algebra generated by $\mathfrak{M}_{\gamma,\mathrm{l}}$ and $\mathfrak{M}_{\gamma,\mathrm{r}}$ and

$$\left\|\prod_{j=1}^{n} A_j - \prod_{j=1}^{n} B_j\right\| \leq c\epsilon.$$

This proves that

$$I \in \left(\mathfrak{M}_{\gamma,\mathrm{l}} \cup \mathfrak{M}_{\gamma,\mathrm{r}}\right)''. \tag{89}$$

Step 3. We have

$$\begin{aligned}
&\left(\mathfrak{M}_{\gamma,\mathrm{l}} \cup \mathfrak{M}_{\gamma,\mathrm{l}}'\right)' \subset \left(\mathfrak{M}_{\gamma,\mathrm{l}} \cup \mathfrak{M}_{\gamma,\mathrm{r}}\right)' = \left(\mathfrak{M}_{\gamma,\mathrm{l}} \cup \mathfrak{M}_{\gamma,\mathrm{r}} \cup \{I\}\right)' \\
&= \{\phi((1-\chi)^{\frac{1}{2}} w_1 - \chi^{\frac{1}{2}} w_2, \overline{\chi}^{\frac{1}{2}}\overline{w}_1 + (1-\chi)^{\frac{1}{2}}\overline{w}_2) \ : \ w_1, w_2 \in \mathcal{Z}\}' \\
&= \{\phi(w_1, \overline{w}_2) \ : \ w_1, w_2 \in \mathcal{Z}\}' = \mathbb{C}1,
\end{aligned}$$

where at the beginning we used Step 1, then (89), next we used

$$\phi_{\gamma,\mathrm{r}}(\overline{z}) = \mathrm{i}\phi(-\chi^{\frac{1}{2}}\mathrm{i}z, -(1-\chi)^{\frac{1}{2}}\overline{\mathrm{i}z})I,$$

$$\{(1-\chi)^{\frac{1}{2}}w_1 - \chi^{\frac{1}{2}}w_2, \overline{\chi}^{\frac{1}{2}}\overline{w}_1 + (1-\chi)^{\frac{1}{2}}\overline{w}_2) \ : \ w_1, w_2 \in \mathcal{Z}\} = \mathcal{Z} \oplus \overline{\mathcal{Z}},$$

and finally the irreducibility of fermionic fields. This shows that $\big(\mathfrak{M}_{\gamma,\mathrm{l}} \cup \mathfrak{M}_{\gamma,\mathrm{l}}'\big)' = \mathbb{C}1$, which means that $\mathfrak{M}_{\gamma,\mathrm{l}}$ is a factor and ends the proof of 5).

The proof of the $\Rightarrow$ part of 6) is similar to its bosonic analog. Assume that $\mathrm{Ker}\gamma = \{0\}$. Set

$$\tau^t(A) := \Gamma(\gamma, \overline{\gamma}^{-1})^{\mathrm{i}t} A \Gamma(\gamma, \overline{\gamma}^{-1})^{-\mathrm{i}t}.$$

We first check that τ^t preserves $\mathfrak{M}_{\gamma,\mathrm{l}}$. Therefore, it is a W^*-dynamics on $\mathfrak{M}_{\gamma,\mathrm{l}}$.

Next we check that $(\Omega| \cdot \Omega)$ satisfies the $(\tau, -1)$-KMS condition. This is straightforward for the Weyl operators $\phi_{\gamma,\mathrm{l}}(z)$. Therefore, it holds for the $*-$algebra $\mathfrak{M}_{\gamma,\mathrm{l},0}$ of polynomials in $\phi_{\gamma,\mathrm{l}}(z)$. By the Kaplansky Theorem, the unit ball of $\mathfrak{M}_{\gamma,\mathrm{l},0}$ is σ-weakly dense in the unit ball of $\mathfrak{M}_{\gamma,\mathrm{l}}$. Using this we extend the KMS condition to $\mathfrak{M}_{\gamma,\mathrm{l}}$.

A KMS state on a factor is always faithful. By 5), $\mathfrak{M}_{\gamma,\mathrm{l}}$ is a factor. Hence Ω is separating.

Let $\mathcal{H}$ be the closure of $\mathfrak{M}_{\gamma,\mathrm{l}}\Omega$. $\mathcal{H}$ is invariant for $\mathfrak{M}_{\gamma,\mathrm{l}}$, moreover Ω is cyclic and separating for $\mathfrak{M}_{\gamma,\mathrm{l}}$ on $\mathcal{H}$. Computing on polynomials in $\phi_{\gamma,\mathrm{l}}(z)$ acting on Ω, we check that $\Gamma(\gamma, \overline{\gamma}^{-1})$ and J_{a} preserve $\mathcal{H}$, the modular conjugation for Ω is given by $J_{\mathrm{a}}\Big|_{\mathcal{H}}$ and the modular operator equals $\Gamma(\gamma, \overline{\gamma}^{-1})\Big|_{\mathcal{H}}$. Now

$$\begin{aligned} &\phi_{\gamma,\mathrm{l}}(z_1)\cdots\phi_{\gamma,\mathrm{l}}(z_n) J_{\mathrm{a}} \phi_{\gamma,\mathrm{l}}(z_1')\cdots\phi_{\gamma,\mathrm{l}}(z_m')\Omega \\ &= \phi_{\gamma,\mathrm{l}}(z_1)\cdots\phi_{\gamma,\mathrm{l}}(z_n)\phi_{\gamma,\mathrm{r}}(\overline{z}_1')\cdots\phi_{\gamma,\mathrm{r}}(\overline{z}_m')\Omega \end{aligned}$$

Thus, $\mathfrak{M}_{\gamma,\mathrm{l}} J_{\mathrm{a}} \mathfrak{M}_{\gamma,\mathrm{l}}\Omega$ is dense in $\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. But $\mathfrak{M}_{\gamma,\mathrm{l}} J_{\mathrm{a}} \mathfrak{M}_{\gamma,\mathrm{l}}\Omega \subset \mathcal{H}$. Hence Ω is cyclic and $\mathcal{H} = \Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. This proves 6).

In this section, we will prove 7) only under the assumption $\mathrm{Ker}\gamma = \mathrm{Ker}\gamma^{-1} = \{0\}$. By 5) this implies that $(\Omega| \cdot \Omega)$ is faithful and we can apply the modular theory, which gives $J_{\mathrm{a}}\mathfrak{M}_{\gamma,\mathrm{l}}J_{\mathrm{a}} = \mathfrak{M}_{\gamma,\mathrm{l}}'$. By 3) we have $J_{\mathrm{a}}\mathfrak{M}_{\gamma,\mathrm{l}}J_{\mathrm{a}} = \mathfrak{M}_{\gamma,\mathrm{r}}$.

7) for a general γ will follow from Theorem 55, proven later. □

10.3 Quasi-free representations of the CAR as the Araki-Wyss representations

There is a simple condition, which allows to check whether a given quasi-free representation of the CAR is unitarily equivalent to an Araki-Wyss representation:

Theorem 44. *Suppose that* $(\mathcal{Y}, \alpha)$ *is a real Hilbert space,*

$$\mathcal{Y} \ni y \mapsto \phi(y) \in B_{\mathrm{h}}(\mathcal{H}) \tag{90}$$

is a representation of the CAR and Ψ *a cyclic quasi-free vector for (90). Let* ω *be defined by (83). Suppose that* $\mathrm{Ker}\omega$ *is even or infinite dimensional.*

Then there exists a complex Hilbert space $\mathcal{Z}$, an operator γ on $\mathcal{Z}$ satisfying $0 \leq \gamma \leq 1$, and an isometric operator $U : \mathcal{H} \to \Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ such that

$$U\Psi = \Omega,$$

$$U\phi(y) = \phi_{\gamma,1}(y)U, \quad y \in \mathcal{Y}.$$

$\mathcal{Z}$ equipped with the real part of its scalar product coincides with $(\mathcal{Y}, \alpha)$.

Proof. By Theorem 42 2), there exists an antisymmetric operator μ such that $\|\mu\| \leq 1$ and

$$y_1\omega y_2 = 2y_1\alpha\mu y_2.$$

Let $\mathcal{Y}_0 := \mathrm{Ker}\mu$ and $\mathcal{Y}_1$ be its orthogonal complement. On $\mathcal{Y}_1$ we can make the polar decomposition

$$\mu = |\mu|j = j|\mu|,$$

j is an orthogonal operator such that $j^2 = -1$. Thus j is a complex structure. If $\dim \mathcal{Y}_0$ is even or infinite, then we can extend j to a complex structure on $\mathcal{Y}$. Interpreting $-j$ as the complex structure, we convert $\mathcal{Y}$ into a complex space, which will be denoted by $\mathcal{Z}$, and equip it with the (sesquilinear) scalar product

$$(y_1|y_2) := y_1\alpha y_2 + \mathrm{i}y_1\alpha j y_2.$$

Now $\chi := \frac{1}{2}(1 - |\mu|)$ and $\gamma := \chi(1-\chi)^{-1}$ define operators on $\mathcal{Z}$ such that $0 \leq \chi \leq \frac{1}{2}$ and $0 \leq \gamma \leq 1$. We have

$$\begin{aligned}
(\Psi|\phi(y_1)\phi(y_2)\Psi) &= y_1\alpha y_2 + \tfrac{\mathrm{i}}{2}y_1\omega y_2 \\
&= y_1\alpha y_2 + \mathrm{i}y_1\alpha j y_2 + \mathrm{i}y_1\alpha j(|\mu| - 1)y_2 \\
&= (y_1|y_2) - 2\mathrm{i}\mathrm{Im}(y_1|\chi y_2) \\
&= (\Omega|\phi_{\gamma,1}(y_1)\phi_{\gamma,1}(y_2)\Omega).
\end{aligned}$$

Now we see that

$$U\phi(y)\Psi := \phi_{\gamma,1}(y)\Omega$$

extends to an isometric operator intertwining the representation (90) with the Araki-Wyss representation for $(\mathcal{Z}, \gamma)$. □

10.4 Tracial quasi-free representations

Tracial representations of the CAR are the fermionic analogs of classical quasifree representations of the CCR.

Consider a real Hilbert space $\mathcal{V}$. Let $\mathcal{W} = \mathcal{V} \oplus \mathrm{i}\mathcal{V}$ be its complexification. Let κ denote the natural conjugation on $\mathcal{W}$, which means $\kappa(v_1 + \mathrm{i}v_2) := v_1 - \mathrm{i}v_2$, $v_1, v_2 \in \mathcal{V}$. Consider the pair of representations of the CAR

$$\begin{aligned} \mathcal{V} \ni v &\mapsto \phi_{\mathcal{V},\mathrm{l}}(v) := \phi(v) \in B_{\mathrm{h}}(\Gamma_{\mathrm{a}}(\mathcal{W})), \\ \mathcal{V} \ni v &\mapsto \phi_{\mathcal{V},\mathrm{r}}(v) := \Lambda\phi(v)\Lambda \in B_{\mathrm{h}}(\Gamma_{\mathrm{a}}(\mathcal{W})). \end{aligned} \tag{91}$$

Let $\mathfrak{M}_{\mathcal{V},\mathrm{l}}$ and $\mathfrak{M}_{\mathcal{V},\mathrm{r}}$ be the von Neumann algebras generated by $\{\phi_{\mathcal{V},\mathrm{l}}(v) \ : \ v \in \mathcal{V}\}$ and $\{\phi_{\mathcal{V},\mathrm{r}}(v) \ : \ v \in \mathcal{V}\}$ respectively.

Theorem 45. 1) *(91) are two commuting representations of the CAR:*

$$\begin{aligned} &[\phi_{\mathcal{V},\mathrm{l}}(v_1), \phi_{\mathcal{V},\mathrm{l}}(v_2)]_+ = 2(v_1|v_2), \\ &[\phi_{\mathcal{V},\mathrm{r}}(v_1), \phi_{\mathcal{V},\mathrm{r}}(v_2)]_+ = 2(v_1|v_2), \\ &[\phi_{\mathcal{V},\mathrm{l}}(v_1), \phi_{\mathcal{V},\mathrm{r}}(v_2)] = 0. \end{aligned}$$

2) *Set*

$$J_{\mathrm{a}} := \Lambda\Gamma(\kappa). \tag{92}$$

We have

$$J_{\mathrm{a}}\phi_{\mathcal{V},\mathrm{l}}(v)J_{\mathrm{a}} = \phi_{\mathcal{V},\mathrm{r}}(v).$$

3) *Ω is a quasi-free vector for (91) with the 2-point function*

$$(\Omega|\phi_{\mathcal{V},\mathrm{l}}(v_1)\phi_{\mathcal{V},\mathrm{l}}(v_2)\Omega) = (v_1|v_2).$$

4) *Ω is cyclic and separating on $\mathfrak{M}_{\mathcal{V},\mathrm{l}}$. $(\Omega|\cdot\,\Omega)$ is tracial, which means*

$$(\Omega|AB\Omega) = (\Omega|BA\Omega), \quad A, B \in \mathfrak{M}_{\mathcal{V},\mathrm{l}}.$$

The corresponding modular conjugation is given by (92) and the modular operator equals $\Delta = 1$.

5) *We have $\mathfrak{M}_{\mathcal{V},\mathrm{r}} = \mathfrak{M}'_{\mathcal{V},\mathrm{l}}$.*

6) *If $\dim\mathcal{V}$ is even or infinite, then the tracial representations of the CAR are unitarily equivalent to the Araki-Wyss representations with $\gamma = 1$ (equivalently, $\chi = \frac{1}{2}$).*

7) *If $\dim\mathcal{V}$ is odd, then the center of $\mathfrak{M}_{\mathcal{V},\mathrm{l}}$ is 2-dimensional: it is spanned by 1 and Q introduced in Theorem 9.*

10.5 Putting together an Araki-Wyss and a tracial representation

A general quasifree representation of the CAR can be obtained by putting together an Araki-Wyss representation and a tracial representation. Actually, one can restrict oneself to a tracial representation with just one dimensional $\mathcal{V}$, but we will consider the general case.

Let $\mathcal{Z}$, γ be as in the subsection on Araki-Wyss representations and $\mathcal{V}, \mathcal{W}$ be as in the subsection on tracial representations. Define the following operators on $\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}} \oplus \mathcal{W})$

$$\begin{aligned} \mathcal{Z} \oplus \mathcal{V} \ni (z, v) &\mapsto \phi^{\mathrm{AW}}_{\mathcal{V},\gamma,\mathrm{l}}(z, v) := \phi\big((1-\chi)^{\frac{1}{2}}z, \overline{\chi}^{\frac{1}{2}}\overline{z}, v\big), \\ \overline{\mathcal{Z}} \oplus \mathcal{V} \ni (\overline{z}, v) &\mapsto \phi^{\mathrm{AW}}_{\mathcal{V},\gamma,\mathrm{r}}(\overline{z}, v) := \Lambda\phi\big(\chi^{\frac{1}{2}}z, (1-\overline{\chi})^{\frac{1}{2}}\overline{z}, v\big)\Lambda, \end{aligned} \tag{93}$$

(We drop the superscript AW in what follows).

Theorem 46. 1) *(93) are two commuting representations of the CAR:*

$$[\phi_{\mathcal{V},\gamma,\mathrm{l}}(z_1,v_1),\phi_{\mathcal{V},\gamma,\mathrm{l}}(z_2,v_2)]_+ = 2\mathrm{Re}(z_1|z_2)+2(v_1|v_2),$$

$$[\phi_{\mathcal{V},\gamma,\mathrm{r}}(\overline{z}_1,v_1),\phi_{\mathcal{V},\gamma,\mathrm{r}}(\overline{z}_2,v_2)]_+ = 2\mathrm{Re}(z_1|z_2)+2(v_1|v_2),$$

$$[\phi_{\mathcal{V},\gamma,\mathrm{l}}(z_1,v_1),\phi_{\mathcal{V},\gamma,\mathrm{r}}(\overline{z}_2,v_2)] = 0.$$

2) *Set*

$$J_{\mathrm{a}} := \Lambda\Gamma(\epsilon\oplus\kappa). \tag{94}$$

We have

$$J_{\mathrm{a}}\phi_{\mathcal{V},\gamma,\mathrm{l}}(v)J_{\mathrm{a}} = \phi_{\mathcal{V},\gamma,\mathrm{r}}(v).$$

3) *Ω is a quasi-free vector for with the 2-point function*

$$(\Omega|\phi_{\mathcal{V},\gamma,\mathrm{l}}(z_1,v_1)\phi_{\mathcal{V},\gamma,l}(z_2,v_2)\Omega) = (z_1|z_2)-2\mathrm{iIm}(z_1|\chi z_2)+(v_1|v_2).$$

4) $\mathrm{Ker}\gamma = \mathrm{Ker}\gamma^{-1} = \{0\}$ *(equivalently,* $\mathrm{Ker}\chi = \mathrm{Ker}(1-\chi) = \{0\}$*) iff Ω is separating on $\mathfrak{M}_{\mathcal{V},\gamma,\mathrm{l}}$ iff Ω is cyclic on $\mathfrak{M}_{\mathcal{V},\gamma,\mathrm{l}}$. If this is the case, the corresponding modular conjugation is given by (94) and the modular operator equals $\Delta = \Gamma(\gamma\oplus\overline{\gamma}^{-1}\oplus 1)$.*

5) $\mathfrak{M}_{\mathcal{V},\gamma,\mathrm{r}} = \mathfrak{M}'_{\mathcal{V},\gamma,\mathrm{l}}$.

6) *If* $\dim\mathcal{V}$ *is even or infinite, then the representations (93) are unitarily equivalent to the Araki-Wyss representations.*

7) *If* $\dim\mathcal{Z}$ *is finite and* $\dim\mathcal{V}$ *is odd, then the center of $\mathfrak{M}_{\mathcal{V},\mathrm{l}}$ is 2-dimensional: it is spanned by* 1 *and Q introduced in Theorem 9.*

11 Confined Bose and Fermi gas

Sometimes the Araki-Woods representation of the CCR and the Araki-Wyss representations of the CAR are equivalent to a multiple of the Fock representation and the corresponding W^*-algebra is type I. This happens e.g. in the case of a finite number of degrees of freedom. More generally, this holds if

$$\mathrm{Tr}\gamma < \infty. \tag{95}$$

Representations satisfying this condition will be called “confined”.

Let us explain the name “confined”. Consider free Bose or Fermi gas with the Hamiltonian equal to $\mathrm{d}\Gamma(h)$, where h is the 1-particle Hamiltonian. One can argue that in the physical description of this system stationary quasi-free states are of special importance. They are given by density matrices of the form

$$\Gamma(\gamma)/\mathrm{Tr}\Gamma(\gamma), \tag{96}$$

with γ commuting with h. In particular, γ can have the form $\mathrm{e}^{-\beta h}$, in which case (96) is the Gibbs state at inverse temperature β.

For (96) to make sense, $\mathrm{Tr}\Gamma(\gamma)$ has to be finite. As we will see later on, $\mathrm{Tr}\Gamma(\gamma) < \infty$ is equivalent to (95).

A typical 1-particle Hamiltonian h of free Bose or Fermi gas is the Laplacian with, say, Dirichlet boundary conditions at the boundary of its domain. If the domain is unbounded, then, usually, the spectrum of h is continuous and, therefore, there are no non-zero operators γ that commute with h and satisfy (95). If the domain is bounded ("confined"), then the spectrum of h is discrete, and hence many such operators γ exist. In particular, $\gamma = \mathrm{e}^{-\beta h}$ has this property. This is the reason why we call "confined" the free Bose or Fermi gas satisfying (95).

In this section we will show how Araki-Woods and Araki-Wyss representations arise in confined systems. We will construct a natural intertwiner between the Araki-Woods/Araki-Wyss representations and the Fock representation. We will treat the bosonic and fermionic case parallel. Whenever possible, we will use the same formula to describe both the bosonic and fermionic case. Some of the symbols will denote different things in the bosonic/fermionic cases (e.g. the fields $\phi(z)$); others will have subscripts s/a indicating the two possible meanings. Sometimes there will be signs $\pm$ or $\mp$ indicating the two possible versions of the formula, the upper in the bosonic case, the lower in the fermionic case.

11.1 Irreducible representation

In this subsection we consider the W^*-algebra $B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$ acting in the obvious way on the Hilbert space $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$. Recall that the W^*-algebra $B(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ is generated by the representation of the CCR

$$\mathcal{Z} \ni z \mapsto W(z) = \mathrm{e}^{\mathrm{i}\phi(z)} \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$$

and the W^*-algebra $B(\Gamma_{\mathrm{a}}(\mathcal{Z}))$ is generated by the representation of CAR

$$\mathcal{Z} \ni z \mapsto \phi(z) \in B_{\mathrm{h}}(\Gamma_{\mathrm{a}}(\mathcal{Z})).$$

In both bosonic and fermionic cases we will also use a certain operator γ on $\mathcal{Z}$.

Recall that in the bosonic case, γ satisfies $0 \leq \gamma \leq 1$, $\mathrm{Ker}(1-\gamma) = \{0\}$, and we introduce the 1-particle density operator denoted ρ, as in (73).

In the fermionic case, γ is a positive operator, possibly with a non-dense domain, and we introduce the 1-particle density denoted χ, as in (84).

Throughout the section we assume that γ is trace class. In the bosonic case it is equivalent to assuming that ρ is trace class. We have

$$\mathrm{Tr}\Gamma(\gamma) = \det(1-\gamma)^{-1} = \det(1+\rho).$$

In the fermionic case, if we assume that $\mathrm{Ker}\gamma^{-1} = \{0\}$ (or $\mathrm{Ker}(\chi - 1) = \{0\}$), γ is trace class iff χ is trace class. We have

$$\mathrm{Tr}\Gamma(\gamma) = \det(1+\gamma) = \det(1-\chi)^{-1}.$$

Define the state ω_γ on the W^*-algebra $B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$ given by the density matrix

$$\Gamma(\gamma)/\mathrm{Tr}\Gamma(\gamma).$$

Let h be another self-adjoint operator on $\mathcal{Z}$. Define the dynamics on $B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$:

$$\tau^t(A) := \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)} A \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(h)} \qquad A \in B(\Gamma_{\mathrm{s}}(\mathcal{Z})).$$

Clearly, ω_γ is τ-invariant iff h commutes with γ.

The state ω_γ is (β,τ)-KMS iff γ is proportional to $\mathrm{e}^{-\beta h}$.

11.2 Standard representation

We need to identify the complex conjugate of the Fock space $\overline{\Gamma_{\mathrm{s/a}}(\mathcal{Z})}$ with the Fock space over the complex conjugate $\Gamma_{\mathrm{s/a}}(\overline{\mathcal{Z}})$. In the bosonic case this is straightforward. In the fermionic case, however, we will not use the naive identification, but the identification that "reverses the order of particles", consistent with the convention adopted in Subsection 2.3. More precisely, if $z_1,\dots,z_n \in \mathcal{Z}$, then the identification looks as follows:

$$\begin{aligned}\overline{\Gamma_{\mathrm{a}}^n(\mathcal{Z})} \ni \overline{z_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} z_n} \mapsto \ & V\overline{z_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} z_n} \\ := \ & \overline{z}_n \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} \overline{z}_1 \in \Gamma_{\mathrm{a}}^n(\overline{\mathcal{Z}}).\end{aligned} \tag{97}$$

(Thus the identification $V : \overline{\Gamma_{\mathrm{a}}(\mathcal{Z})} \to \Gamma_{\mathrm{a}}(\overline{\mathcal{Z}})$ equals Λ times the naive, "non-reversing", identification).

Using (97) at the second step and the exponential law at the last step, we have the identification

$$\begin{aligned} B^2(\Gamma_{\mathrm{s/a}}(\mathcal{Z})) &\simeq \Gamma_{\mathrm{s/a}}(\mathcal{Z}) \otimes \overline{\Gamma_{\mathrm{s/a}}(\mathcal{Z})} \\ &\simeq \Gamma_{\mathrm{s/a}}(\mathcal{Z}) \otimes \Gamma_{\mathrm{s/a}}(\overline{\mathcal{Z}}) \simeq \Gamma_{\mathrm{s/a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}).\end{aligned} \tag{98}$$

As before, define

$$J_{\mathrm{s}} := \Gamma(\epsilon), \quad J_{\mathrm{a}} := \Lambda\Gamma(\epsilon).$$

Theorem 47. *In the bosonic/fermionic case, under the above identification, the hermitian conjugation $*$ becomes $J_{\mathrm{s/a}}$.*

Proof. We restrict ourselves to the fermionic case. Consider

$$B = |z_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} z_n)(w_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} w_m| \in B^2(\Gamma_{\mathrm{a}}(\mathcal{Z})).$$

It corresponds to

$$\begin{aligned}&\sqrt{(n+m)!}z_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} z_n \otimes_{\mathrm{a}} \overline{w_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} w_m}\\ &= \sqrt{(n+m)!}z_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} z_n \otimes_{\mathrm{a}} \overline{w}_m \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} \overline{w}_1 \in \Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}).\end{aligned}$$

On the other hand,

$$B^* = |w_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} w_m)(z_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} z_n|$$

corresponds to

$$\begin{aligned}&\sqrt{(n+m)!}w_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} w_m \otimes_{\mathrm{a}} \overline{z_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} z_n}\\ &= \sqrt{(n+m)!}w_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} w_m \otimes_{\mathrm{a}} \overline{z}_n \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} \overline{z}_1\\ &= (-1)^{\frac{n(n-1)}{2}+\frac{m(m-1)}{2}+nm}\sqrt{(n+m)!}\overline{z}_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} \overline{z}_n \otimes_{\mathrm{a}} w_m \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} w_1\\ &= \Lambda\Gamma(\epsilon)\sqrt{(n+m)!}z_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} z_n \otimes_{\mathrm{a}} \overline{w}_m \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} \overline{w}_1\end{aligned}$$

where at the last step we used $\Gamma(\epsilon)z_i = \overline{z}_i$, $\Gamma(\epsilon)\overline{w}_i = w_i$ and

$$\tfrac{n(n-1)}{2} + \tfrac{m(m-1)}{2} + nm = \tfrac{(n+m)(n+m-1)}{2}.$$

□

The W^*-algebras $B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$ and $\overline{B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))}$ have a natural standard representation in the Hilbert space $B^2(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$, as described in Subsection 8.5. They have also a natural representation in the Hilbert space $\Gamma_{\mathrm{s/a}}(\mathcal{Z})\otimes\overline{\Gamma_{\mathrm{s/a}}(\mathcal{Z})}$, as described in Subsection 8.6. Using the identification (98) we obtain the representation θ_{l} of $B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$ and θ_{r} of $\overline{B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))}$ in the space $\Gamma_{\mathrm{s/a}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$.

Let us describe the last representation in detail. Let

$$U : \Gamma_{\mathrm{s/a}}(\mathcal{Z}) \otimes \Gamma_{\mathrm{s/a}}(\overline{\mathcal{Z}}) \to \Gamma_{\mathrm{s/a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) \tag{99}$$

be the unitary map defined as in (36). Let V be defined in (97). Then

$$\begin{aligned}B(\Gamma_{\mathrm{s/a}}(\mathcal{Z})) \ni A \mapsto \theta_{\mathrm{l}}(A) &:= U\, A\otimes 1_{\Gamma_{\mathrm{s/a}}(\overline{\mathcal{Z}})}\, U^* \in B(\Gamma_{\mathrm{s/a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})),\\ \overline{B(\Gamma_{\mathrm{s}}(\mathcal{Z}))} \ni \overline{A} \mapsto \theta_{\mathrm{r}}(\overline{A}) &:= U\, 1_{\Gamma_{\mathrm{s}}(\mathcal{Z})}\otimes\overline{A}\, U^* \in B(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})),\\ \overline{B(\Gamma_{\mathrm{a}}(\mathcal{Z}))} \ni \overline{A} \mapsto \theta_{\mathrm{r}}(\overline{A}) &:= U\, 1_{\Gamma_{\mathrm{a}}(\mathcal{Z})}\otimes(V\overline{A}V^*)\, U^* \in B(\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})).\end{aligned} \tag{100}$$

We have 2 commuting representations of the CCR

$$\mathcal{Z} \ni z \mapsto W(z,0) = \theta_{\mathrm{l}}(W(z)) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})), \tag{101}$$

$$\overline{\mathcal{Z}} \ni \overline{z} \mapsto W(0,\overline{z}) = \theta_{\mathrm{r}}(\overline{W(z)}) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})), \tag{102}$$

The algebra $\theta_{\mathrm{l}}(B(\Gamma_{\mathrm{s}}(\mathcal{Z})))$ is generated by the image of (101) and the algebra $\theta_{\mathrm{r}}(B(\Gamma_{\mathrm{s}}(\overline{\mathcal{Z}})))$ is generated by the image of (102).

We have also 2 commuting representations of the CAR

$$\mathcal{Z} \ni z \mapsto \phi(z,0) = \theta_{\mathrm{l}}(\phi(z)) \in B_{\mathrm{h}}(\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})), \tag{103}$$

$$\overline{\mathcal{Z}} \ni \overline{z} \mapsto \Lambda\phi(0,\overline{z})\Lambda = \theta_{\mathrm{r}}(\overline{\phi(z)}) \in B_{\mathrm{h}}(\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})). \tag{104}$$

The algebra $\theta_{\mathrm{l}}(B(\Gamma_{\mathrm{a}}(\mathcal{Z})))$ is generated by the image of (103) and the algebra $\theta_{\mathrm{r}}(B(\Gamma_{\mathrm{a}}(\overline{\mathcal{Z}})))$ is generated by the image of (104).

Let $\Gamma_{\mathrm{s/a}}^{+}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ be the image of $B_{+}^{2}(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$ under the identification (98).

Theorem 48. 1) $\left(\theta_{\mathrm{l}}, \Gamma_{\mathrm{s/a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}), J_{\mathrm{s/a}}, \Gamma_{\mathrm{s/a}}^{+}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\right)$ *is a standard representation of* $B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$.

2) $J_{\mathrm{s/a}}\theta_{\mathrm{l}}(A)J_{\mathrm{s/a}} = \theta_{\mathrm{r}}(\overline{A})$.

3) $\mathrm{d}\Gamma(h\oplus(-\overline{h}))$ *is the standard Liouvillean of* $t\mapsto\tau^{t}$ *in this representation*

4) *The standard vector representative of* ω_γ *in this representation is*

$$\Omega_\gamma := \det(1\mp\gamma)^{\pm\frac{1}{2}} \exp\left(\tfrac{1}{2}a^{*}\left(\begin{bmatrix} 0 & \gamma^{\frac{1}{2}} \\ \pm\overline{\gamma}^{\frac{1}{2}} & 0 \end{bmatrix}\right)\right)\Omega. \tag{105}$$

Proof. 1), 2) and 3) are straightforward. Let us prove 4), which is a little involved, since we have to use various identifications we have introduced.

In the representation of Subsection 8.5, the standard vector representative of ω_γ equals

$$\begin{aligned} &(\mathrm{Tr}\Gamma(\gamma))^{-\frac{1}{2}}\Gamma(\gamma^{\frac{1}{2}}) \\ &= (\mathrm{Tr}\Gamma(\gamma))^{-\frac{1}{2}}\sum_{n=0}^{\infty}\Theta_{\mathrm{s/a}}^{n}(\gamma^{\frac{1}{2}})^{\otimes n}\Theta_{\mathrm{s/a}}^{n} \in B^{2}(\Gamma_{\mathrm{s/a}}(\mathcal{Z})). \end{aligned}$$

(Recall that $\Theta_{\mathrm{s/a}}^{n}$ denotes the orthogonal projection onto $\Gamma_{\mathrm{s/a}}^{n}(\mathcal{Z})$).

Clearly, $\gamma^{1/2}\in B^{2}(\mathcal{Z})$ corresponds to a certain vector $\Psi\in\mathcal{Z}\otimes\overline{\mathcal{Z}}$.

Let σ be the permutation of $(1,\dots,2n)$ given by

$$\sigma(2j-1)=j,\quad \sigma(2j)=2n-j+1,\quad j=1,\dots,n.$$

This permutation defines the unitary transformation

$$\Theta(\sigma) : (\mathcal{Z}\otimes\overline{\mathcal{Z}})^{\otimes n} \to (\otimes^{n}\mathcal{Z})\otimes\left(\otimes^{n}\overline{\mathcal{Z}}\right).$$

Now $(\gamma^{1/2})^{\otimes n}$ can be interpreted in two fashions. It can be interpreted as an element of $\otimes^{n}B^{2}(\mathcal{Z})$, and then it corresponds to the vector $\Psi^{\otimes n}\in(\mathcal{Z}\otimes\overline{\mathcal{Z}})^{\otimes n}$. It can be also interpreted as an element of $B^{2}(\mathcal{Z}^{\otimes n})$ and then it corresponds to

$$\Theta(\sigma)\Psi^{\otimes n}\in(\otimes^{n}\mathcal{Z})\otimes\left(\otimes^{n}\overline{\mathcal{Z}}\right)\simeq\otimes^{n}\mathcal{Z}\otimes\left(\overline{\otimes^{n}\mathcal{Z}}\right).$$

(Note that we have taken into account the convention about the complex conjugate of the tensor product adopted in Subsect. 2.3).

Now $\Theta_{\mathrm{s/a}}^{n}(\gamma^{1/2})^{\otimes n}\Theta_{\mathrm{s/a}}^{n}\in B^{2}(\Gamma_{\mathrm{s/a}}^{n}(\mathcal{Z}))$ corresponds to

$$\left(\Theta^n_{\rm s/a}\otimes\overline{\Theta^n_{\rm s/a}}\right)\ \Theta(\sigma)\ \Psi^{\otimes n}\in\Gamma^n_{\rm s/a}(\mathcal{Z})\otimes\Gamma^n_{\rm s/a}(\overline{\mathcal{Z}}). \tag{106}$$

The identification

$$U:\Gamma^n_{\rm s/a}(\mathcal{Z})\otimes\Gamma^n_{\rm s/a}(\overline{\mathcal{Z}})\to\Gamma^{2n}_{\rm s/a}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$$

is obtained by first treating $\Gamma^n_{\rm s/a}(\mathcal{Z})\otimes\Gamma^n_{\rm s/a}(\overline{\mathcal{Z}})$ as a subspace of $\otimes^{2n}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ and then applying $\frac{\sqrt{(2n)!}}{n!}\Theta^{2n}_{\rm s/a}$. Therefore, (106) is identified with

$$\begin{aligned}&\frac{\sqrt{(2n)!}}{n!}\Theta^{2n}_{\rm s/a}\ \left(\Theta^n_{\rm s/a}\otimes\overline{\Theta^n_{\rm s/a}}\right)\ \Theta(\sigma)\ \Psi^{\otimes n}\\ =&\frac{\sqrt{(2n)!}}{n!}\Theta^{2n}_{\rm s/a}\left(\Theta^2_{\rm s/a}\Psi\right)^{\otimes n}=\frac{\sqrt{(2n)!}}{n!}\left(\Theta^2_{\rm s/a}\Psi\right)^{\otimes_{\rm s/a}n}\in\Gamma^n_{\rm s/a}(\mathcal{Z}\oplus\overline{\mathcal{Z}}),\end{aligned} \tag{107}$$

where we used the fact that

$$\begin{aligned}\Theta^{2n}_{\rm s/a}\ \left(\Theta^n_{\rm s/a}\otimes\overline{\Theta^n_{\rm s/a}}\right)\ \Theta(\sigma)&=\Theta^{2n}_{\rm s/a}\\ &=\Theta^{2n}_{\rm s/a}\ \left(\Theta^2_{\rm s/a}\otimes\cdots\otimes\Theta^2_{\rm s/a}\right).\end{aligned} \tag{108}$$

(In the fermionic case, to see the first identity of (108) we need to note that the permutation σ is even). Now, if τ denotes the transposition, then $\Theta^2_{\rm s/a}\Psi=\frac12(\Psi\pm\Theta(\tau)\Psi)$. Recall that $\Psi\in\mathcal{Z}\otimes\overline{\mathcal{Z}}$ corresponds to $\gamma^{1/2}\in B^2(\mathcal{Z})$. Therefore, $\Theta(\tau)\Psi$ corresponds to $(\gamma^{1/2})^{\#}=\overline{\gamma}^{1/2}$. Hence, $\Theta^2_{\rm s/a}\Psi\in\Gamma^2_{\rm s/a}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ is identified with

$$\frac12 c=\frac12\begin{bmatrix}0&\gamma^{\frac12}\\ \pm\overline{\gamma}^{\frac12}&0\end{bmatrix}\in B^2_{\rm s/a}(\overline{\mathcal{Z}}\oplus\mathcal{Z},\mathcal{Z}\oplus\overline{\mathcal{Z}}). \tag{109}$$

Thus, (107) corresponds to

$$(n!)^{-1}\left(\frac12 a^*(c)\right)^n\Omega\in\Gamma^{2n}_{\rm s/a}(\mathcal{Z}\oplus\overline{\mathcal{Z}}).$$

So, finally, $\Gamma(\gamma^{\frac12})$ corresponds to

$$\exp\left(\tfrac12 a^*(c)\right)\Omega.$$

Clearly,

$$cc^*=\begin{bmatrix}0&\gamma^{\frac12}\\ \pm\overline{\gamma}^{\frac12}&0\end{bmatrix}\begin{bmatrix}0&\pm\overline{\gamma}^{\frac12}\\ \gamma^{\frac12}&0\end{bmatrix}=\begin{bmatrix}\gamma&0\\0&\overline{\gamma}\end{bmatrix}.$$

Therefore,

$$\det(1\mp cc^*)^{\mp\frac12}=\det(1\mp\gamma)^{\mp1}={\rm Tr}\Gamma(\gamma).$$

□

Note that the vector Ω_γ is an example of a bosonic/fermionic Gaussian state considered in (43) and (55), where it was denoted Ω_c:

$$\Omega_\gamma=\det(1\mp cc^*)^{\pm\frac14}\exp(\tfrac12 a^*(c))\Omega.$$

11.3 Standard representation in the Araki-Woods/Araki-Wyss form

Define the following transformation on $\Gamma_{\mathrm{s/a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$:

$$\begin{aligned} R_\gamma := &\det(1 \mp \gamma)^{\pm\frac{1}{2}} \exp\left(\mp\tfrac{1}{2}a^*\left(\left[\begin{array}{cc} 0 & \gamma^{\frac{1}{2}} \\ \pm\overline{\gamma}^{\frac{1}{2}} & 0 \end{array}\right]\right)\right) \\ &\times\Gamma\big((1\mp\gamma)\oplus(1\mp\overline{\gamma})\big)^{\pm\frac{1}{2}} \exp\left(\pm\tfrac{1}{2}a\left(\left[\begin{array}{cc} 0 & \gamma^{\frac{1}{2}} \\ \pm\overline{\gamma}^{\frac{1}{2}} & 0 \end{array}\right]\right)\right). \end{aligned} \tag{110}$$

Theorem 49. R_γ *is a unitary operator satisfying*

$$\begin{aligned} &R_\gamma \phi(z_1, \overline{z_2}) R_\gamma^* \\ &= \phi((1\mp\gamma)^{\pm\frac{1}{2}} z_1 \pm (\gamma \mp 1)^{\pm\frac{1}{2}} z_2, (\overline{\gamma} \mp 1)^{\pm\frac{1}{2}} \overline{z_1} + (1 \mp \overline{\gamma})^{\pm\frac{1}{2}} \overline{z_2}). \end{aligned} \tag{111}$$

Proof. Let c be defined as in (109). Using

$$\Gamma(1 \mp cc^*) = \Gamma((1\mp\gamma) \oplus (1 \mp \overline{\gamma})),$$

we see that

$$R_\gamma := \det(1 \mp cc^*)^{\pm\frac{1}{4}} \exp(\mp\tfrac{1}{2}a^*(c)) \Gamma(1 \mp cc^*)^{\pm\frac{1}{2}} \exp(\pm\tfrac{1}{2}a(c)).$$

Thus R_γ is in fact the transformation R_c considered in (51) and (62). □

Let $\phi_{\gamma,\mathrm{l}}(z)$, $\phi_{\gamma,\mathrm{r}}(z)$, $\Gamma^+_{\mathrm{s/a},\gamma}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$, etc. be defined as in Theorems 40 and 43.

R_γ intertwines between the usual left/right representations and the left/right Araki-Woods/Araki-Wyss representations, which is expressed by the following identities:

$$\begin{aligned} R_\gamma \phi(z, 0)) R_\gamma^* &= \phi_{\gamma,\mathrm{l}}(z), \\ R_\gamma \phi(0, \overline{z}) R_\gamma^* &= \phi_{\gamma,\mathrm{r}}(\overline{z}), \text{ in the bosonic case,} \\ R_\gamma \Lambda \phi(0, \overline{z}) \Lambda R_\gamma^* &= \phi_{\gamma,\mathrm{r}}(\overline{z}), \text{ in the fermionic case,} \\ R_\gamma \theta_\mathrm{l}(B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))) R_\gamma^* &= \mathfrak{M}_{\gamma,\mathrm{l}}, \\ R_\gamma \theta_\mathrm{r}(\overline{B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))}) R_\gamma^* &= \mathfrak{M}_{\gamma,\mathrm{r}}, \\ R_\gamma J_{\mathrm{s/a}} R_\gamma^* &= J_{\mathrm{s/a}}, \\ R_\gamma \Gamma^+_{\mathrm{s/a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) &= \Gamma^+_{\mathrm{s/a},\gamma}(\mathcal{Z} \oplus \overline{\mathcal{Z}}), \\ R_\gamma \Omega_\gamma &= \Omega, \\ R_\gamma \mathrm{d}\Gamma(h, -\overline{h}) R_\gamma^* &= \mathrm{d}\Gamma(h, -\overline{h}). \end{aligned}$$

For $A \in B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$, set

$$\begin{aligned}\theta_{\gamma,\mathrm{l}}(A) &:= R_\gamma \theta_{\mathrm{l}}(A) R_\gamma^* \in B(\Gamma_{\mathrm{s/a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})),\\ \theta_{\gamma,\mathrm{r}}(\overline{A}) &:= R_\gamma \theta_{\mathrm{r}}(\overline{A}) R_\gamma^* \in B(\Gamma_{\mathrm{s/a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})).\end{aligned} \tag{112}$$

Finally, we see that in the confined case the algebra $\mathfrak{M}_{\gamma,\mathrm{l}}$ is isomorphic to $B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$:

Theorem 50. 1) $\left(\theta_{\gamma,\mathrm{l}}, \Gamma_{\mathrm{s/a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}), J_{\mathrm{s/a}}, \Gamma^+_{\mathrm{s/a},\gamma}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\right)$ *is a standard representation of* $B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$.
2) $J_{\mathrm{s/a}} \theta_{\gamma,\mathrm{l}}(A) J_{\mathrm{s/a}} = \theta_{\gamma,\mathrm{r}}(\overline{A})$.
3) $\mathrm{d}\Gamma(h \oplus (-\overline{h}))$ *is the standard Liouvillean of* $t \mapsto \tau^t$ *in this representation.*
4) Ω *is the standard vector representative of* ω_γ *in this representation.*

12 Lattice of von Neumann algebras in a Fock space

Let $\mathcal{Z}$ be a Hilbert space. With every real closed subspace of $\mathcal{Z}$ we can naturally associate a certain von Neumann subalgebra of $B(\Gamma_{\mathrm{s/a}}(\mathcal{Z}))$, both in the bosonic and fermionic case. These von Neumann subalgebras form a complete lattice. Properties of this lattice are studied in this section. They have important applications in quantum field theory.

12.1 Real subspaces in a complex Hilbert space

In this subsection we analyze real subspaces in a complex Hilbert space. (For a similar analysis of two complex subspaces in a complex Hilbert space see [28, 43].) This analysis will be then used both in the bosonic and fermionic case. We start with a simple fact which is true both in the complex and real case.

Lemma 1. *Let* $\mathcal{V}_1, \mathcal{V}_2$ *be closed subspaces of a (real or complex) Hilbert space. Let* p_1, p_2 *be the corresponding orthogonal projections. Then*

$$\mathcal{V}_1 \cap \mathcal{V}_2 + \mathcal{V}_1^\perp \cap \mathcal{V}_2^\perp = \mathrm{Ker}(p_1 - p_2).$$

Proof. Let $(p_1 - p_2)z = 0$. Then $z = p_1 z + (1-p_1)z$, where $p_1 z = p_2 z \in \mathcal{V}_1 \cap \mathcal{V}_2$ and $(1-p_1)z = (1-p_2)z \in \mathcal{V}_1^\perp \cap \mathcal{V}_2^\perp$. □

Next suppose that $\mathcal{W}$ is a complex Hilbert space. Then it is at the same time a real Hilbert space with the scalar product given by the real part of the original scalar product. If $K \subset \mathcal{W}$, then $K^\perp$ will denote the orthogonal complement of K in the sense of the complex scalar product and K^{perp} will denote the orthogonal complement with respect to the real scalar product. That means

$$K^{\rm perp} := \{z \in \mathcal{Z} \ : \ {\rm Re}(v|z) = 0, \ v \in K\}.$$

Moreover,

$$\mathrm{i}K^{\rm perp} = \{z \in \mathcal{Z} \ : \ {\rm Im}(v|z) = 0, \ v \in K\},$$

so $\mathrm{i}K^{\rm perp}$ can be called the symplectic complement of K. Note that if $\mathcal{V}$ is a closed real subspace of $\mathcal{W}$, then $(\mathcal{V}^{\rm perp})^{\rm perp} = \mathcal{V}$ and $\mathrm{i}(\mathrm{i}\mathcal{V}^{\rm perp})^{\rm perp} = \mathcal{V}$.

Theorem 51. *Let $\mathcal{V}$ be a closed real subspace of a complex Hilbert space $\mathcal{W}$. Let p, q be the orthogonal projections onto $\mathcal{V}$ and $\mathrm{i}\mathcal{V}$ respectively. Then the following conditions holds:*
1) $\mathcal{V} \cap \mathrm{i}\mathcal{V} = \mathcal{V}^{\rm perp} \cap \mathrm{i}\mathcal{V}^{\rm perp} = \{0\} \ \Leftrightarrow \ {\rm Ker}(p-q) = \{0\}$*;*
2) $\mathcal{V}^{\rm perp} \cap \mathrm{i}\mathcal{V} = \{0\} \Leftrightarrow \ {\rm Ker}(p+q-1) = \{0\}$*.*

Proof. By the previous lemma applied to the real Hilbert space $\mathcal{W}$ and its subspaces $\mathcal{V}$, $\mathrm{i}\mathcal{V}$ we get

$$\mathcal{V} \cap \mathrm{i}\mathcal{V} + \mathcal{V}^{\rm perp} \cap \mathrm{i}\mathcal{V}^{\rm perp} = {\rm Ker}(p-q).$$

This gives 1). Applying this lemma to $\mathcal{V}$, $\mathrm{i}\mathcal{V}^{\rm perp}$ yields

$$\mathcal{V} \cap \mathrm{i}\mathcal{V}^{\rm perp} + \mathcal{V}^{\rm perp} \cap \mathrm{i}\mathcal{V} = {\rm Ker}(p+q-1).$$

Using $\mathcal{V} \cap \mathrm{i}\mathcal{V}^{\rm perp} = \mathrm{i}(\mathcal{V}^{\rm perp} \cap \mathrm{i}\mathcal{V})$ we obtain 2). □

We will say that a real subspace $\mathcal{V}$ of a complex Hilbert space is *in a general position* if it satisfies both conditions of the previous theorem. The following fact is immediate:

Theorem 52. *Let $\mathcal{V}$ be a closed real subspace of a complex Hilbert space $\mathcal{W}$. Set*

$$\mathcal{W}_+ := \mathcal{V} \cap \mathrm{i}\mathcal{V}, \qquad \mathcal{W}_- := \mathcal{V}^{\rm perp} \cap \mathrm{i}\mathcal{V}^{\rm perp},$$

$$\mathcal{W}_1 := \mathcal{V} \cap \mathrm{i}\mathcal{V}^{\rm perp} + \mathrm{i}\mathcal{V} \cap \mathcal{V}^{\rm perp}, \quad \mathcal{W}_0 := (\mathcal{W}_+ + \mathcal{W}_- + \mathcal{W}_0)^\perp,$$

Then $\mathcal{W}_-, \mathcal{W}_+, \mathcal{W}_0, \mathcal{W}_1$ are complex subspaces of $\mathcal{W}$ and

$$\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_-.$$

We have

$$\mathcal{W}_+ = \mathcal{V} \cap \mathcal{W}_+, \quad \{0\} = \mathcal{V} \cap \mathcal{W}_-.$$

Set

$$\mathcal{V}_0 := \mathcal{V} \cap \mathcal{W}_0, \qquad \mathcal{V}_1 := \mathcal{V} \cap \mathcal{W}_1 = \mathcal{V} \cap \mathrm{i}\mathcal{V}^{\rm perp}.$$

Then $\mathcal{V}_0$ is a subspace of $\mathcal{W}_0$ in a general position and

$$\mathcal{V} = \mathcal{W}_+ \oplus \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \{0\},$$

$$\mathrm{i}\mathcal{V}^{\rm perp} = \{0\} \oplus \mathrm{i}\mathcal{V}_0^{\rm perp} \oplus \mathcal{V}_1 \oplus \mathcal{W}_-,$$

(where $\mathcal{V}_0^{\rm perp}$ is the real orthogonal complement of $\mathcal{V}_0$ taken inside $\mathcal{W}_0$).

Theorem 53. *Let $\mathcal{V}$ be a closed real subspace of a complex Hilbert space $\mathcal{W}$ in a general position. Then the following is true:*

1) *There exists a closed complex subspace $\mathcal{Z}$ of $\mathcal{W}$, a unitary antilinear operator ϵ on $\mathcal{W}$ and a self-adjoint operator χ such that $\epsilon^2 = 1$, $\epsilon\mathcal{Z} = \mathcal{Z}^\perp$, $0 \le \chi \le \frac{1}{2}$, $\mathrm{Ker}\chi = \mathrm{Ker}(\chi - \frac{1}{2}) = \{0\}$ and*

$$\{(1-\chi)^{\frac{1}{2}} z + \epsilon\chi^{\frac{1}{2}} z \ : \ z \in \mathcal{Z}\} = \mathcal{V},$$

$$\{\chi^{\frac{1}{2}} z + \epsilon(1-\chi)^{\frac{1}{2}} z \ : \ z \in \mathcal{Z}\} = \mathrm{i}\mathcal{V}^{\mathrm{perp}}.$$

2) *Set $\rho := \chi(1-2\chi)^{-1}$. Then ρ is a positive operator on $\mathcal{Z}$ with $\mathrm{Ker}\rho = \{0\}$ and*

$$\{(1+\rho)^{\frac{1}{2}} z + \epsilon\rho^{\frac{1}{2}} z \ : \ z \in \mathrm{Dom}\rho^{1/2}\} \quad \textit{is dense in} \ \ \mathcal{V},$$

$$\{\rho^{\frac{1}{2}} z + \epsilon(1+\rho)^{\frac{1}{2}} z \ : \ z \in \mathrm{Dom}\rho^{1/2}\} \quad \textit{is dense in} \ \ \mathrm{i}\mathcal{V}^{\mathrm{perp}}.$$

Proof. Let p, q be defined as above. Clearly, $q = \mathrm{i}p\mathrm{i}^{-1}$.

Define the self-adjoint real-linear operators $m := p + q - 1$, $n := p - q$. Note that

$$n^2 = 1 - m^2 = p + q - pq - qp,$$

$$mn = -nm \quad = qp - pq,$$

$$\begin{aligned}
&\mathrm{i}m = m\mathrm{i}, && \mathrm{i}n = -n\mathrm{i},\\
&\mathrm{Ker}\, m = \{0\}, && \mathrm{Ker}\, n = \{0\},\\
&\mathrm{Ker}(m \pm 1) = \{0\}, && \mathrm{Ker}(n \pm 1) = \{0\},\\
&-1 \le m \le 1, && -1 \le n \le 1.
\end{aligned}$$

We can introduce their polar decompositions

$$n = |n|\epsilon = \epsilon|n|, \quad m = w|m| = |m|w.$$

Clearly, ϵ and w are orthogonal operators satisfying

$$w^2 = \epsilon^2 = 1, \quad w\epsilon = -\epsilon w,$$

$$w\mathrm{i} = \mathrm{i}w, \quad \mathrm{i}\epsilon = -\mathrm{i}\epsilon.$$

Set

$$\mathcal{Z} := \mathrm{Ker}(w - 1) = \mathrm{Ran}1_{]0,1[}(m).$$

Let $1_{\mathcal{Z}}$ denote the orthogonal projection from $\mathcal{W}$ onto $\mathcal{Z}$.

We have

$$\epsilon\mathcal{Z} = \mathrm{Ker}(w + 1) = \mathrm{Ran}1_{]-\infty,0[}(m),$$

Clearly, we have the orthogonal direct sum $\mathcal{W} = \mathcal{Z} \oplus \epsilon\mathcal{Z}$.

Using $p = \frac{m+n+1}{2}$ we get

$$1_{\mathcal{Z}} p 1_{\mathcal{Z}} = \tfrac{m+1}{2} 1_{\mathcal{Z}},$$

$$\epsilon 1_{\mathcal{Z}} \epsilon p 1_{\mathcal{Z}} = \epsilon 1_{\mathcal{Z}} \epsilon \tfrac{n}{2} 1_{\mathcal{Z}} = \epsilon \tfrac{\sqrt{1-m^2}}{2} 1_{\mathcal{Z}}.$$

Therefore,

$$p 1_{\mathcal{Z}} = \tfrac{m+1}{2} 1_{\mathcal{Z}} + \epsilon \tfrac{\sqrt{1-m^2}}{2} 1_{\mathcal{Z}}.$$

Set $\chi := \frac{1}{2} 1_{\mathcal{Z}} (1-m)$. Then

$$\{(1-\chi)^{\frac{1}{2}} z + \epsilon \chi^{\frac{1}{2}} z \ : \ z \in \mathcal{Z}\} = \{pz \ : \ z \in \mathcal{Z}\} \subset \mathcal{V}. \tag{113}$$

Suppose now that $v \in \mathcal{V} \cap \{pz \ : \ z \in \mathcal{Z}\}^{\mathrm{perp}}$. Then

$$0 = \mathrm{Re}(v|p1_{\mathcal{Z}} v) = \mathrm{Re}(v|1_{\mathcal{Z}} v) = \|1_{\mathcal{Z}} v\|^2.$$

Hence, $v \in \mathcal{Z}^{\perp} = \epsilon \mathcal{Z}$. Therefore, using $q = m + 1 - p$ we obtain

$$\mathrm{Re}(v|qv) = \mathrm{Re}(v|mv) \leq 0.$$

Hence $qv = 0$. Thus $v \in \mathrm{i}\mathcal{V}^{\mathrm{perp}} \cap \mathcal{V}$, which means that $v = 0$. Therefore, the left hand side of (113) is dense in $\mathcal{V}$.

To see that we have an equality in (113), we note that the operator

$$(1-\chi)^{1/2} 1_{\mathcal{Z}} + \epsilon \chi^{1/2} 1_{\mathcal{Z}} \tag{114}$$

is an isometry from $\mathcal{Z}$ to $\mathcal{W}$. Hence the range of (114) is closed. This ends the proof of 1).

2) follows easily from 1). □

Note that $\epsilon \mathcal{Z}$ can be identified with $\overline{\mathcal{Z}}$. Thus $\mathcal{W}$ can be identified with $\mathcal{Z} \oplus \overline{\mathcal{Z}}$. Under this identification, the operator ϵ coincides with ϵ defined in (75).

Theorem 53 gives 2 descriptions of a real subspace $\mathcal{V}$. The description 2) is used in the Araki-Woods representations of the CCR and 1) in the Araki-Wyss representations of the CAR.

12.2 Complete lattices

In this subsection we recall some definitions concerning abstract lattices (see e.g. [53]). They provide a convenient language that can be used to express some properties of a class of von Neumann algebras acting on a Fock space.

Suppose that $(X, \leq)$ is an ordered set. Let $\{x_i \ : \ i \in I\}$ be a nonempty subset of X.

We say that u is a largest minorant of $\{x_i \ : \ i \in I\}$ if

1) $i \in I$ implies $u \leq x_i$;
2) $u_1 \leq x_i$ for all $i \in I$ implies $u_1 \leq u$.

If $\{x_i \ : \ i \in I\}$ possesses a largest minorant, then it is uniquely defined. The largest minorant of a set $\{x_i \ : \ i \in I\}$ is usually denoted

$$\mathop{\wedge}_{i \in I} x_i.$$

Analogously we define the smallest majorant of $\{x_i \ : \ i \in I\}$, which is usually denoted by

$$\mathop{\vee}_{i \in I} x_i.$$

We say that $(X, \leq)$ is a complete lattice, if every nonempty subset of X possesses the largest minorant and the smallest majorant. It is then equipped with the operations $\wedge$ and $\vee$.

We will say that the complete lattice is complemented if it is equipped with the operation $X \ni x \mapsto \sim x \in X$ such that

1) $\sim (\sim x) = x$;
2) $x_1 \leq x_2$ implies $\sim x_2 \leq \sim x_1$;
3) $\sim \mathop{\wedge}_{i \in I} x_i = \mathop{\vee}_{i \in I} (\sim x_i)$.

The operation $\sim$ will be called the complementation.

Let $\mathcal{W}$ be a topological vector space. For a family $\{\mathcal{V}_i\}_{i \in I}$ of closed subspaces of $\mathcal{W}$ we define

$$\mathop{\vee}_{i \in I} \mathcal{V}_i := \left(\sum_{i \in I} \mathcal{V}_i \right)^{\mathrm{cl}}.$$

Closed subspaces of $\mathcal{W}$ form a complete lattice with the order relation $\subset$ and the operations $\cap$ and $\vee$. If in addition $\mathcal{W}$ is a (real or complex) Hilbert space, then taking the orthogonal complement is an example of a complementation. In the case of a complex Hilbert space (or a finite dimensional symplectic space), taking the symplectic complement is also an example of a complementation.

Let $\mathcal{H}$ be a Hilbert space. For a family of von Neumann algebras $\mathfrak{M}_i \subset B(\mathcal{H})$, $i \in I$, we set

$$\mathop{\vee}_{i \in I} \mathfrak{M}_i := \left(\bigcup_{i \in I} \mathfrak{M}_i \right)''.$$

(Recall that the prime denotes the commutant). Von Neumann algebras in $B(\mathcal{H})$ form a complete lattice with the order relation $\subset$ and the operations $\cap$ and $\vee$. Taking the commutant is an example of a complementation.

12.3 Lattice of von Neumann algebras in a bosonic Fock space

In this subsection we describe the result of Araki describing the lattice of von Neumann algebras naturally associated to a bosonic Fock space [1, 32]. In the proof of this result it is convenient to use the facts about the Araki-Woods representation derived earlier.

Let $\mathcal{W}$ be a complex Hilbert space. We will identify $\mathrm{Re}(\mathcal{W}\oplus\overline{\mathcal{W}})$ with $\mathcal{W}$. Consider the Hilbert space $\Gamma_{\mathrm{s}}(\mathcal{W})$ and the corresponding Fock representation $\mathcal{W}\ni w\mapsto W(w)\in B(\Gamma_{\mathrm{s}}(\mathcal{W}))$.

For a real subspace $\mathcal{V}\subset\mathcal{W}$ we define the von Neumann algebra

$$\mathfrak{M}(\mathcal{V}) := \{W(w)\ :\ w\in\mathcal{V}\}'' \subset B(\Gamma_{\mathrm{s}}(\mathcal{W})).$$

First note that it follows from the strong continuity of $\mathcal{W}\ni w\mapsto W(w)$ that $\mathfrak{M}(\mathcal{V})=\mathfrak{M}(\mathcal{V}^{\mathrm{cl}})$. Therefore, in what follows it is enough to restrict ourselves to closed subspaces of $\mathcal{W}$.

The following theorem was proven by Araki [1], and then a simpler proof of the most difficult statement, the duality (6), was given by Eckmann and Osterwalder [32]:

Theorem 54. 1) $\mathfrak{M}(\mathcal{V}_1)=\mathfrak{M}(\mathcal{V}_2)$ *iff* $\mathcal{V}_1=\mathcal{V}_2$.
2) $\mathcal{V}_1\subset\mathcal{V}_2$ *implies* $\mathfrak{M}(\mathcal{V}_1)\subset\mathfrak{M}(\mathcal{V}_2)$.
3) $\mathfrak{M}(\mathcal{W})=B(\Gamma_{\mathrm{s}}(\mathcal{W}))$ *and* $\mathfrak{M}(\{0\})=\mathbb{C}1$.
4) $\mathfrak{M}\Big(\mathop{\vee}\limits_{i\in I}\mathcal{V}_i\Big)=\mathop{\vee}\limits_{i\in I}\mathfrak{M}(\mathcal{V}_i)$.
5) $\mathfrak{M}\Big(\bigcap\limits_{i\in I}\mathcal{V}_i\Big)=\bigcap\limits_{i\in I}\mathfrak{M}(\mathcal{V}_i)$.
6) $\mathfrak{M}(\mathcal{V})'=\mathfrak{M}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})$.
7) $\mathfrak{M}(\mathcal{V})$ *is a factor iff* $\mathcal{V}\cap\mathrm{i}\mathcal{V}^{\mathrm{perp}}=\{0\}$.

Proof. To prove 1), assume that $\mathcal{V}_1$ and $\mathcal{V}_2$ are distinct closed subspaces. It is enough to assume that $\mathcal{V}_2\not\subset\mathcal{V}_1$. Then we can find $w\in\mathrm{i}\mathcal{V}_1^{\mathrm{perp}}\backslash\mathrm{i}\mathcal{V}_2^{\mathrm{perp}}$. Now $W(w)\in\mathfrak{M}(\mathcal{V}_1)'\backslash\mathfrak{M}(\mathcal{V}_2)'$. This implies $\mathfrak{M}(\mathcal{V}_1)'\neq\mathfrak{M}(\mathcal{V}_2)'$, which yields (1).

2) and 3) are immediate. The inclusion $\subset$ in 4) and the inclusion $\supset$ in 5) are immediate. The inclusion $\supset$ in 4) follows easily if we invoke the strong continuity of $\mathcal{W}\ni w\mapsto W(w)$.

If we know 6), then the remaining inclusion $\subset$ in 5) follows from $\supset$ in 4). 7) follows from 1), 5) and 6).

Thus what remains to be shown is (6). Its original proof was surprisingly involved, see [1]. We will give a somewhat simpler proof [32], which uses properties of the Araki-Woods representations, which in turn are based on the Tomita-Takesaki theory.

First, assume that $\mathcal{V}$ is in general position in $\mathcal{W}$. Then, according to Theorem 53 2), and identifying $\epsilon\mathcal{Z}$ with $\overline{\mathcal{Z}}$, we obtain a decomposition $\mathcal{V}=\mathcal{Z}\oplus\overline{\mathcal{Z}}$ and a positive operator ρ such that

$$\{(1+\rho)^{\frac{1}{2}}z+\overline{\rho}^{\frac{1}{2}}\overline{z}\ :\ z\in\mathcal{Z}\}\ \text{ is dense in }\ \mathcal{V}.$$

Then we see that $\mathfrak{M}(\mathcal{V})$ is the left Araki-Woods algebra $\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$. By Theorem 40, the commutant of $\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$ is $\mathfrak{M}_{\rho,\mathrm{r}}^{\mathrm{AW}}$. But

$$\{\rho^{\frac{1}{2}}z+(1+\overline{\rho})^{\frac{1}{2}}\overline{z}\ :\ z\in\mathcal{Z}\}\ \text{ is dense in }\ \mathrm{i}\mathcal{V}^{\mathrm{perp}}.$$

Therefore, $\mathfrak{M}_{\rho,\mathrm{r}}^{\mathrm{AW}}$ coincides with $\mathfrak{M}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})$. This ends the proof of 6) in the case of $\mathcal{V}$ in a general position.

For an arbitrary $\mathcal{V}$, we decompose $\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_-$ and $\mathcal{V} = \mathcal{W}_+ \oplus \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \{0\}$, as in Theorem 52. Then we can write

$$B(\Gamma_{\mathrm{s}}(\mathcal{W})) \simeq B(\Gamma_{\mathrm{s}}(\mathcal{W}_+)) \otimes B(\Gamma_{\mathrm{s}}(\mathcal{W}_0)) \otimes B(\Gamma_{\mathrm{s}}(\mathcal{W}_1)) \otimes B(\Gamma_{\mathrm{s}}(\mathcal{W}_-)),$$

$$\mathfrak{M}(\mathcal{V}) \simeq B(\Gamma_{\mathrm{s}}(\mathcal{W}_+)) \otimes \mathfrak{M}(\mathcal{V}_0) \otimes \mathfrak{M}(\mathcal{V}_1) \otimes 1.$$

Clearly, $\mathrm{i}\mathcal{V}^{\mathrm{perp}} = \{0\} \oplus \mathrm{i}\mathcal{V}_0^{\mathrm{perp}} \oplus \mathcal{V}_1 \oplus \mathcal{W}_-$ and the commutant of $\mathfrak{M}(\mathcal{V})$ equals

$$\begin{aligned}\mathfrak{M}(\mathcal{V})' &\simeq 1 \otimes \mathfrak{M}(\mathcal{V}_0)' \otimes \mathfrak{M}(\mathcal{V}_1) \otimes B(\Gamma_{\mathrm{s}}(\mathcal{W}_-)) \\ &= 1 \otimes \mathfrak{M}(\mathrm{i}\mathcal{V}_0^{\mathrm{perp}}) \otimes \mathfrak{M}(\mathcal{V}_1) \otimes B(\Gamma_{\mathrm{s}}(\mathcal{W}_-)) \\ &\simeq \mathfrak{M}(\mathrm{i}\mathcal{V}^{\mathrm{perp}}).\end{aligned}$$

□

Note that the above theorem can be interpreted as an isomorphism of the complete lattices of closed real subspaces of the complex Hilbert space $\mathcal{W}$ with the symplectic complement as the complementation, and the lattice of von Neumann algebras $\mathfrak{M}(\mathcal{V}) \subset B(\Gamma_{\mathrm{s}}(\mathcal{W}))$, with the complementation given by the commutant.

12.4 Lattice of von Neumann algebras in a fermionic Fock space

In this subsection we describe the fermionic analog of Araki's result about the lattice of von Neumann algebras in a bosonic Fock space.

Again, consider a complex Hilbert space $\mathcal{W}$. We will identify $\mathrm{Re}(\mathcal{W} \oplus \overline{\mathcal{W}})$ with $\mathcal{W}$. Consider the Hilbert space $\Gamma_{\mathrm{a}}(\mathcal{W})$ and the corresponding Fock representation $\mathcal{W} \ni w \mapsto \phi(w) \in B(\Gamma_{\mathrm{a}}(\mathcal{W}))$.

Consider the Hilbert space $\Gamma_{\mathrm{a}}(\mathcal{W})$ and the corresponding Fock representation. We will identify $\mathrm{Re}(\mathcal{W} \oplus \overline{\mathcal{W}})$ with $\mathcal{W}$. For a real subspace $\mathcal{V} \subset \mathcal{W}$ we define the von Neumann algebra

$$\mathfrak{M}(\mathcal{V}) := \{\phi(z) \ : \ z \in \mathcal{V}\}'' \subset B(\Gamma_{\mathrm{a}}(\mathcal{W})).$$

Let the operator Λ defined in (37).

First note that it follows from the norm continuity of $\mathcal{W} \ni w \mapsto \phi(w)$ that $\mathfrak{M}(\mathcal{V}) = \mathfrak{M}(\mathcal{V}^{\mathrm{cl}})$. Therefore, in what follows it is enough to restrict ourselves to closed real subspaces of $\mathcal{W}$.

Theorem 55. 1) $\mathfrak{M}(\mathcal{V}_1) = \mathfrak{M}(\mathcal{V}_2)$ *iff* $\mathcal{V}_1 = \mathcal{V}_2$.
2) $\mathcal{V}_1 \subset \mathcal{V}_2$ *implies* $\mathfrak{M}(\mathcal{V}_1) \subset \mathfrak{M}(\mathcal{V}_2)$.
3) $\mathfrak{M}(\mathcal{W}) = B(\Gamma_{\mathrm{a}}(\mathcal{W}))$ *and* $\mathfrak{M}(\{0\}) = \mathbb{C}1$.
4) $\mathfrak{M}\Big(\bigcap\limits_{i\in I} \mathcal{V}_i\Big) = \bigcap\limits_{i\in I} \mathfrak{M}(\mathcal{V}_i)$.

5) $\mathfrak{M}\Big(\underset{i\in I}{\vee}\mathcal{V}_i\Big) = \underset{i\in I}{\vee}\mathfrak{M}(\mathcal{V}_i)$.
6) $\mathfrak{M}(\mathcal{V})' = \Lambda\mathfrak{M}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})\Lambda$.

The proof of the above theorem is very similar to the proof of Theorem 54 from the bosonic case. The main additional difficulty is the behavior of fermionic fields under the tensor product. They are studied in the following theorem.

Theorem 56. *Let $\mathcal{W}_i$, $i = 1,2$ be two Hilbert spaces and $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ Let N_i be the number operators in $\Gamma_\mathrm{a}(\mathcal{Z}_i)$, $i = 1,2$, $I_i := (-1)^{N_i}$ and $\Lambda_i := (-1)^{N_i(N_i-1)/2}$. We identify the operators on $\Gamma_\mathrm{a}(\mathcal{W})$ with those on $\Gamma_\mathrm{a}(\mathcal{W}_1)\otimes\Gamma_\mathrm{a}(\mathcal{W}_2)$ using U defined in (36). This identification is denoted by $\simeq$. Let $\mathcal{V}_i$, $i = 1,2$ be real closed subspaces of $\mathcal{W}_i$, $i = 1,2$ resp. Then*

$$\mathfrak{M}(\mathcal{V}_1 \oplus \mathcal{V}_2) \simeq \big(\mathfrak{M}(\mathcal{V}_1)\otimes 1 + (-1)^{N_1\otimes N_2} 1\otimes\mathfrak{M}(\mathcal{V}_2)(-1)^{N_1\otimes N_2}\big)'', \tag{115}$$

$$\begin{aligned} \mathfrak{M}(\mathcal{V}_1 \oplus \{0\}) &\simeq \mathfrak{M}(\mathcal{V}_1)\otimes 1, \\ \mathfrak{M}(\mathcal{W}_1 \oplus \mathcal{V}_2) &\simeq B(\Gamma_\mathrm{a}(\mathcal{W}_1))\otimes\mathfrak{M}(\mathcal{V}_2). \end{aligned} \tag{116}$$

$$\begin{aligned} \Lambda\mathfrak{M}(\mathcal{V}_1 \oplus \mathcal{W}_2)\Lambda &\simeq \Lambda_1\mathfrak{M}(\mathcal{V}_1)\Lambda_1\otimes B(\Gamma_\mathrm{a}(\mathcal{W}_2)), \\ \Lambda\mathfrak{M}(\{0\} \oplus \mathcal{V}_2)\Lambda &\simeq\ 1\otimes\Lambda_2\mathfrak{M}(\mathcal{V}_2)\Lambda_2\ . \end{aligned} \tag{117}$$

Proof. Let $v\in\mathcal{V}_2$. Then, by Theorem 19 2), we have the identification

$$\phi(0,v) \simeq (-1)^{N_1}\otimes\phi(v) = (-1)^{N_1\otimes N_2}\ 1\otimes\phi(v)\ (-1)^{N_1\otimes N_2}.$$

Therefore, the von Neumann algebra generated by $\phi(0,v)$, $v\in\mathcal{V}_2$ equals $(-1)^{N_1\otimes N_2}1\otimes\mathfrak{M}(\mathcal{V}_2)(-1)^{N_1\otimes N_2}$.

Clearly, the von Neumann algebra generated by $\phi(v,0)$, $v\in\mathcal{V}_1$ equals $\mathfrak{M}(\mathcal{V}_1)\otimes 1$. This implies (115).

(115) implies immediately (116). It also implies

$$\begin{aligned} \mathfrak{M}(\mathcal{V}_1 \oplus \mathcal{W}_2) &\simeq (-1)^{N_1\otimes N_2}\mathfrak{M}(\mathcal{V}_1)\otimes B(\Gamma_\mathrm{a}(\mathcal{W}_2))(-1)^{N_1\otimes N_2}, \\ \mathfrak{M}(\{0\} \oplus \mathcal{V}_2) &\simeq (-1)^{N_1\otimes N_2}\ 1\otimes\mathfrak{M}(\mathcal{V}_2)\ (-1)^{N_1\otimes N_2}, \end{aligned} \tag{118}$$

from which (117) follows by Theorem 20 1). □

Proof of Theorem 55. Let us first prove 1). Assume that $\mathcal{V}_1$ and $\mathcal{V}_2$ are distinct closed subspaces. It is enough to assume that $\mathcal{V}_2 \not\subset \mathcal{V}_1$. Then we can find $w\in\mathrm{i}\mathcal{V}_1^{\mathrm{perp}}\backslash\mathrm{i}\mathcal{V}_2^{\mathrm{perp}}$. Now $\Lambda\phi(w)\Lambda\in\mathfrak{M}(\mathcal{V}_1)'\backslash\mathfrak{M}(\mathcal{V}_2)'$. Thus $\mathfrak{M}(\mathcal{V}_1)'\neq\mathfrak{M}(\mathcal{V}_2)'$, which yields 1).

Similarly as in the proof of Theorem 54 the only difficult part is a proof of 6).

Assume first that $\mathcal{V}$ satisffies $\mathcal{V}\cap\mathrm{i}\mathcal{V} = \mathcal{V}^{\mathrm{perp}}\cap\mathrm{i}\mathcal{V}^{\mathrm{perp}} = \{0\}$. By Theorem 52, we can write $\mathcal{W} = \mathcal{W}_0\oplus\mathcal{W}_1$ and $\mathcal{V} = \mathcal{V}_0\oplus\mathcal{V}_1$ where $\mathcal{V}_0$ is in general

position in $\mathcal{W}_0$ and $\mathcal{V}_1^{\mathrm{perp}} = \mathrm{i}\mathcal{V}_1$ inside $\mathcal{W}_1$. By Theorem 53, we can find a complex subspace $\mathcal{Z}$ of $\mathcal{W}_0$, an antilinear involution ϵ on $\mathcal{W}_0$ and a self-adjoint operator $0 \leq \chi \leq \frac{1}{2}$ such that $\epsilon\mathcal{Z} = \mathcal{Z}^\perp$, $\mathrm{Ker}\chi = \mathrm{Ker}(\chi - \frac{1}{2}) = \{0\}$ and

$$\{(1-\chi)^{\frac{1}{2}} z + \epsilon\chi^{\frac{1}{2}} z \ : \ z \in \mathcal{Z}\} \oplus \mathcal{V}_1 = \mathcal{V},$$

$$\{\chi^{\frac{1}{2}} z + \epsilon(1-\chi)^{\frac{1}{2}} z \ : \ z \in \mathcal{Z}\} \oplus \mathcal{V}_1 = \mathrm{i}\mathcal{V}^{\mathrm{perp}}.$$

We can identify $\epsilon\mathcal{Z}$ with $\overline{\mathcal{Z}}$, using ϵ as the conjugation. Then we are precisely in the framework of Theorem 46, which implies that $\mathfrak{M}(\mathcal{V})' = \Lambda\mathfrak{M}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})\Lambda$.

For an arbitrary $\mathcal{V}$, to decompose $\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_-$ and $\mathcal{V} = \mathcal{W}_+ \oplus \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \{0\}$ as in Theorem 52. Then we can write

$$B(\Gamma_{\mathrm{a}}(\mathcal{W})) \simeq B(\Gamma_{\mathrm{a}}(\mathcal{W}_+)) \otimes B(\Gamma_{\mathrm{a}}(\mathcal{W}_0 \oplus \mathcal{W}_1)) \otimes B(\Gamma_{\mathrm{a}}(\mathcal{W}_-)),$$

$$\mathfrak{M}(\mathcal{V}) \simeq B(\Gamma_{\mathrm{a}}(\mathcal{W}_+)) \otimes \mathfrak{M}(\mathcal{V}_0 \oplus \mathcal{V}_1) \otimes 1$$

Let N_{01} be the number operator on $\Gamma_{\mathrm{a}}(\mathcal{W}_0 \oplus \mathcal{W}_1)$ and $\Lambda_{01} := (-1)^{N_{01}(N_{01}-1)/2}$. The commutant of $\mathfrak{M}(\mathcal{V})$ equals

$$\begin{aligned}\mathfrak{M}(\mathcal{V})' &\simeq 1 \otimes \mathfrak{M}(\mathcal{V}_0 \oplus \mathcal{V}_1)' \otimes B(\Gamma_{\mathrm{a}}(\mathcal{W}_-)),\\ &= 1 \otimes \Lambda_{01}\mathfrak{M}(\mathrm{i}(\mathcal{V}_0 \oplus \mathcal{V}_1)^{\mathrm{perp}})\Lambda_{01} \otimes B(\Gamma_{\mathrm{a}}(\mathcal{W}_-))\\ &\simeq \Lambda\mathfrak{M}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})\Lambda,\end{aligned}$$

where in the last step we used Theorem 56. □

13 Pauli-Fierz systems

In this section we would like to illustrate the following phenomenon. We start from a certain physically well motivated quantum system describing a small system interacting with the Bose gas. The Hamiltonian that generates the dynamics is a certain self-adjoint operator, bounded from below, partly expressed in terms of the usual creation and annihillation operators.

Suppose that we want to consider the same system in the thermodynamical limit corresponding to a nonzero density ρ (and the corresponding γ defined by (73)). For instance, we are interested in the density given by the Planck law at inverse temperature β. We can do this as follows: we change the representation of the CCR from the original Fock representation to the Araki-Woods representation at γ. We still assume that the dynamics is formally generated by the same expression.

To make this idea rigorous, it is convenient to use the framework of W^*-dynamical systems. In fact, what we obtain is a family of W^*-dynamical systems $(\mathfrak{M}_\gamma, \tau_\gamma)$ depending on γ, in general non-isomorphic to one another.

Even if we fix γ, then we can consider various unitarily non-equivalent representations of the W^*-dynamical system $(\mathfrak{M}_\gamma, \tau_\gamma)$. In fact, in the literature such systems are considered in at least two different representations.

The first one is what we call the semistandard representation. It was used mostly in the older literature, e.g. by Davies [23]. It is quite simple: the small system is assumed to interact with positive density Araki-Woods fields. In this representation, the dynamics has a unitary implementation given by the unitary group generated by, what we call, the semi-Liouvillean.

The second one is the standard representation. It is commonly used in the more recent literature [27, 11]. One can argue that it is the most natural representation from the point of view of theory of W^*-algebras. In any case, it is a useful tool to study various properties of $(\mathfrak{M}_\gamma, \tau_\gamma)$. On the other hand, it is more complicated than the semistandard representation. The natural implementation of the dynamics in this representation is generated by the standard Liouvillean.

If the bosons are confined in the sense of Subsections 11.1, then $(\mathfrak{M}_\gamma, \tau_\gamma)$ are for various γ isomorphic. In this case, the algebra $\mathfrak{M}_\gamma$ has also a third useful representation: the irreducible one, which is not available in the general case.

The main goal of this section is to illustrate the above ideas with the so-called Pauli-Fierz systems. We will use the name *a Pauli-Fierz operator* to denote a self-adjoint operator describing bosons interacting with a small quantum system with an interaction linear in fields. We reserve the name *a Pauli-Fierz Hamiltonian* to Pauli-Fierz operators with a positive dispersion relation. This condition guarantees that they are bounded from below. (Note that Pauli-Fierz Liouvilleans and semi-Liouvilleans are in general not bounded from below).

Pauli-Fierz Hamiltonians arise in quantum physics in various contexts and are known under many names (e.g. the spin-boson Hamiltonian). The Hamiltonian of QED in the dipole approximation is an example of such an operator.

Several aspects of Pauli-Fierz operators have been recently studied in mathematical literature, both because of their physical importance and because of their interesting mathematical properties, see [25, 26, 27, 11] and references therein.

The plan of this section is as follows. First we fix some notation useful in describing small quantum systems interacting with Bose gas (following mostly [27]). Then we describe a Pauli-Fierz Hamiltonian. It is described by a positive boson 1-particle energy h, small system Hamiltonian K and a coupling operator v. (Essentially the only reason to assume that h is positive is the fact that such 1-particle energies are typical for physical systems). The corresponding W^*-algebraic system is just the algebra of all bounded operators on the Hilbert space with the Heisenberg dynamics generated by the Hamiltonian.

Given the operator γ describing the boson fields (and the corresponding operator ρ describing the boson density, related to γ by (73)) we construct the W^*-dynamical system $(\mathfrak{M}_\gamma, \tau_\gamma)$ – the Pauli-Fierz system corresponding to

γ. The system $(\mathfrak{M}_\gamma, \tau_\gamma)$ is described in two representations: the semistandard and the standard one.

Parallel to the general case, we describe the confined case. We show, in particular, that in the confined case the semi-Liouvilleans and Liouvilleans are unitarily equivalent for various densities γ.

The constructions presented in this section are mostly taken from [27]. The only new material is the discussion of the confined case, which, even if straightforward, we believe to be quite instructive.

Remark 4. In all our considerations about Pauli-Fierz systems we restrict ourselves to the W^*-algebraic formalism. It would be tempting to apply the C^*-algebraic approach to describe Pauli-Fierz systems [17]. This approach proposes that a quantum system should be described by a certain C^*-dynamical system (a C^*-algebra with a strongly continuous dynamics). By considering various representations of this C^*-dynamical system one could describe its various thermodynamical behaviors.

Such an approach works usually well in the case of infinitely extended spin or fermionic systems, because in a finite volume typical interactions are bounded [17]), and in algebraic local quantum field theory, because of the finite speed of propagation [40]. Unfortunately, for Pauli-Fierz systems the C^*-approach seems to be inappropriate – we do not know of a good choice of a C^*-algebra with the dynamics generated by a non-trivial Pauli-Fierz Hamiltonian. The problem is due to the unboundedness of bosonic fields that are involved in Pauli-Fierz Hamiltonians.

13.1 Creation and annihilation operators in coupled systems

Suppose that $\mathcal{W}$ is a Hilbert space. Consider a bosonic system described by the Fock space $\Gamma_{\rm s}(\mathcal{W})$ interacting with a quantum system described by a Hilbert space $\mathcal{E}$. The composite system is described by the Hilbert space $\mathcal{E}\otimes\Gamma_{\rm s}(\mathcal{W})$. In this subsection we discuss the formalism that we will use to describe the interaction of such coupled systems.

Let $q\in B(\mathcal{E},\mathcal{E}\otimes\mathcal{W})$. The annihilation operator $a(q)$ is a densely defined operator on $\mathcal{E}\otimes\Gamma_{\rm s}(\mathcal{W})$ with the domain equal to the finite particle subspace of $\mathcal{E}\otimes\Gamma_{\rm s}(\mathcal{W})$. For $\Psi\in\mathcal{E}\otimes\Gamma_{\rm s}^n(\mathcal{W})$ we set

$$a(q)\Psi := \sqrt{n}q^*\otimes 1_{\mathcal{W}}^{\otimes(n-1)}\Psi\in\mathcal{E}\otimes\Gamma_{\rm s}^{n-1}(\mathcal{W}). \tag{119}$$

($\mathcal{E}\otimes\Gamma_{\rm s}^n(\mathcal{W})$ can be viewed as a subspace of $\mathcal{E}\otimes\mathcal{W}^{\otimes n}$. Moreover, $q^*\otimes 1_{\mathcal{W}}^{\otimes(n-1)}$ is an operator from $\mathcal{E}\otimes\mathcal{W}^{\otimes n}$ to $\mathcal{E}\otimes\mathcal{W}^{\otimes(n-1)}$, which maps $\mathcal{E}\otimes\Gamma_{\rm s}^n(\mathcal{W})$ into $\mathcal{E}\otimes\Gamma_{\rm s}^{n-1}(\mathcal{W})$. Therefore, (119) makes sense).

The operator $a(q)$ is closable and we will denote its closure by the same symbol. The creation operator $a^*(q)$ is defined as

$$a^*(q) := a(q)^*.$$

Note that if $q = B \otimes |w)$, for $B \in B(\mathcal{E})$ and $w \in \mathcal{W}$, then

$$a^*(q) = B \otimes a^*(w), \quad a(q) = B^* \otimes a(w),$$

where $a^*(w)/a(w)$ are the usual creation/annihilion operators on the Fock space $\Gamma_{\mathrm{s}}(\mathcal{W})$.

13.2 Pauli-Fierz Hamiltonians

Throughout this section we assume that K is a self-adjoint operator on a finite dimensional Hilbert space $\mathcal{K}$, h is a positive operator on a Hilbert space $\mathcal{Z}$ and $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$. The self-adjoint operator

$$H_{\mathrm{fr}} := K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h)$$

on $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$ will be called a *free Pauli-Fierz Hamiltonian.* The interaction is described by the self-adjoint operator

$$V = a^*(v) + a(v).$$

The operator

$$H := H_{\mathrm{fr}} + V$$

is called a Pauli-Fierz Hamiltonian.

If

$$h^{-\frac{1}{2}} v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z}), \tag{120}$$

then H is self-adjoint on $\mathrm{Dom}(H_{\mathrm{fr}})$ and bounded from below, see e.g [27].

$\big(B(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})), \mathrm{e}^{\mathrm{i}tH} \cdot \mathrm{e}^{-\mathrm{i}tH}\big)$ will be called a *Pauli-Fierz W^*-dynamical system at zero density.*

Remark 5. We will usually drop $1_{\mathcal{K}} \otimes$ in formulas, so that $h^{-\frac{1}{2}} v$ above should be read $(h^{-\frac{1}{2}} \otimes 1_{\mathcal{K}}) v$.

Remark 6. Self-adjoint operators of the form of a Pauli-Fierz Hamiltonian, but without the requirement that the boson energy is positive, will be called Pauli-Fierz operators.

13.3 More notation

In order to describe Pauli-Fierz systems at a positive density in a compact and elegant way we need more notation.

Let $\mathcal{K}$ and $\mathcal{Z}$ be Hilbert spaces. Remember that we assume $\mathcal{K}$ to be finite dimensional. First we introduce a certain antilinear map $\star$ from $B(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$ to $B(\mathcal{K}, \mathcal{K} \otimes \overline{\mathcal{Z}})$.

Let $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$. We define $v^\star \in B(\mathcal{K}, \mathcal{K} \otimes \overline{\mathcal{Z}})$ such that for $\Phi, \Psi \in \mathcal{K}$ and $w \in \mathcal{Z}$,

$$(\Phi \otimes w \mid v\Psi)_{\mathcal{K} \otimes \mathcal{Z}} = (v^\star \Phi | \Psi \otimes \overline{w})_{\mathcal{K} \otimes \overline{\mathcal{Z}}}.$$

It is easy to see that $v^\star$ is uniquely defined. (Note that $\star$ is different from $*$ denoting the Hermitian conjugation).

Remark 7. Given an orthonormal basis $\{w_i \;:\; i \in I\}$ in $\mathcal{Z}$, any $v \in B(\mathcal{K}, \mathcal{K}\otimes\mathcal{Z})$ can be decomposed as

$$v = \sum_{i\in I} B_i \otimes |w_i), \tag{121}$$

where $B_i \in B(\mathcal{K})$, then

$$v^\star := \sum_{i\in I} B_i^* \otimes |\overline{w_i}).$$

Next we introduce the operation $\check{\otimes}$, which can be called *tensoring in the middle*. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces. If $\overline{B} \in B(\overline{\mathcal{K}})$, $A \in B(\mathcal{K}\otimes\mathcal{H}_1, \mathcal{K}\otimes\mathcal{H}_2)$, we define

$$\overline{B}\check{\otimes}A := (\theta^{-1}\otimes 1_{\mathcal{H}_2})\;(\overline{B}\otimes A)\;(\theta\otimes 1_{\mathcal{H}_1}) \in B(\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\mathcal{H}_1, \mathcal{K}\otimes\overline{\mathcal{K}}\otimes\mathcal{H}_2), \tag{122}$$

where $\theta : \mathcal{K}\otimes\overline{\mathcal{K}} \to \overline{\mathcal{K}}\otimes\mathcal{K}$ is defined as $\theta\;\Psi_1\otimes\overline{\Psi}_2 := \overline{\Psi}_2\otimes\Psi_1$.

Remark 8. If $C \in B(\mathcal{K})$, $A \in B(\mathcal{H}_1, \mathcal{H}_2)$, then $\overline{B}\,\check{\otimes}\,(C\otimes A) := C\otimes\overline{B}\otimes A$.

13.4 Pauli-Fierz systems at a positive density

In this subsection we introduce Pauli-Fierz W^*-dynamical systems. They will be the main subject of the remaining part of this section.

Let ρ be a positive operator commuting with h having the interpretation of the *radiation density*. Let γ be the operator related to ρ as in (73). Let $\mathfrak{M}_{\gamma,1}^{\mathrm{AW}} \subset B(\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$ be the left Araki-Woods algebra introduced in Subsection 9.3. The *Pauli-Fierz algebra corresponding to* γ is defined by

$$\mathfrak{M}_\gamma := B(\mathcal{K})\otimes\mathfrak{M}_{\gamma,1}^{\mathrm{AW}}. \tag{123}$$

The identity map

$$\mathfrak{M}_\gamma \to B(\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})) \tag{124}$$

will be called *the semistandard representation of* $\mathfrak{M}_\gamma$. (The bosonic part of (124) is already standard, the part involving $\mathcal{K}$ is not—hence the name).

Proposition 2. *Assume that*

$$(1+\rho)^{1/2}v \in B(\mathcal{K}, \mathcal{K}\otimes\mathcal{Z}). \tag{125}$$

Let

$$V_\gamma := a^*\Big((1+\rho)^{\frac12}v, \overline{\rho}^{\frac12}v^\star\Big) + a\Big((1+\rho)^{\frac12}v, \overline{\rho}^{\frac12}v^\star\Big).$$

Then the operator V_γ is essentially self-adjoint on the space of finite particle vectors and affiliated to $\mathfrak{M}_\gamma$.

The *free Pauli-Fierz semi-Liouvillean* is the self-adjoint operator on $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ defined as

$$L_{\mathrm{fr}}^{\mathrm{semi}} := K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h \oplus (-\overline{h})).$$

The *full Pauli-Fierz semi-Liouvillean corresponding to* γ is

$$L_{\gamma}^{\mathrm{semi}} := L_{\mathrm{fr}}^{\mathrm{semi}} + V_{\gamma}. \tag{126}$$

Let us formulate the following assumption:

$$(1+h)(1+\rho)^{1/2} v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z}). \tag{127}$$

Using Theorem 3.3 of [29] we obtain

Theorem 57. 1)
$$\tau_{\mathrm{fr}}^{t}(A) := \mathrm{e}^{\mathrm{i}tL_{\mathrm{fr}}^{\mathrm{semi}}} A \mathrm{e}^{-\mathrm{i}tL_{\mathrm{fr}}^{\mathrm{semi}}}$$

is a W^**-dynamics on* $\mathfrak{M}_{\gamma}$.

2) *Suppose that (127) holds. Then* $L_{\gamma}^{\mathrm{semi}}$ *is essentially self-adjoint on* $\mathrm{Dom}(L_{\mathrm{fr}}^{\mathrm{semi}}) \cap \mathrm{Dom}(V_{\gamma})$ *and*

$$\tau_{\gamma}^{t}(A) := \mathrm{e}^{\mathrm{i}tL_{\gamma}^{\mathrm{semi}}} A \mathrm{e}^{-\mathrm{i}tL_{\gamma}^{\mathrm{semi}}}$$

is a W^**-dynamics on* $\mathfrak{M}_{\gamma}$.

The pair $\left(\mathfrak{M}_{\gamma}, \tau_{\gamma}\right)$ will be called the *Pauli-Fierz* W^**-dynamical system corresponding to* γ.

13.5 Confined Pauli-Fierz systems—semistandard representation

In this subsection we make the assumption $\mathrm{Tr}\gamma < \infty$. As before, we will call it the confined case.

We can use the identity representation for $B(\mathcal{K})$ and the Araki-Woods representation $\theta_{\gamma,\mathrm{l}}$ for $B(\Gamma_{\mathrm{s}}(\mathcal{Z}))$. Thus we obtain the faithful representation

$$\pi_{\gamma}^{\mathrm{semi}} : B(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})) \to B(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})),$$

which will be called the semistandard representation of $B(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}))$. In other words, $\pi_{\gamma}^{\mathrm{semi}}$ is defined by

$$\pi_{\gamma}^{\mathrm{semi}}(A) = U_{\gamma}^{\mathrm{semi}} A \otimes 1_{\Gamma_{\mathrm{s}}(\overline{\mathcal{Z}})} U_{\gamma}^{\mathrm{semi}*} \quad A \in B(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})),$$

where

$$U_{\gamma}^{\mathrm{semi}} := 1_{\mathcal{K}} \otimes R_{\gamma} U,$$

and U was defined in (36) and R_{γ} in (110).

Theorem 58.

$$\pi_\gamma^{\mathrm{semi}}\left(B\big(\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z})\big)\right)=\mathfrak{M}_\gamma,$$

$$\pi_\gamma^{\mathrm{semi}}\Big(\mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}\Big)=\tau_\gamma^t\left(\pi_\gamma^{\mathrm{semi}}(A)\right),\quad A\in B(\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z})),$$

$$L_\gamma^{\mathrm{semi}}=U_\gamma^{\mathrm{semi}}\big(H\otimes 1_{\Gamma_{\mathrm{s}}(\overline{\mathcal{Z}})}-1_{\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z})}\otimes\mathrm{d}\Gamma(\overline{h})\big)U_\gamma^{\mathrm{semi}*}.$$

Let us stress that in the confined case the semi-Liouvilleans L_γ^{semi} and the W^*-dynamical systems $(\mathfrak{M}_{\gamma,\mathrm{l}},\tau_\gamma)$ are unitarily equivalent for different γ.

13.6 Standard representation of Pauli-Fierz systems

In this subsection we drop the assumption $\mathrm{Tr}\gamma<\infty$ about the confinement of the bosons and we consider the general case again.

Consider the representation

$$\pi:\mathfrak{M}_\gamma\to B(\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$$

defined by

$$\pi(A):=1_{\overline{\mathcal{K}}}\check{\otimes}A,\quad A\in\mathfrak{M}_\gamma,$$

where $\check{\otimes}$ was introduced in (122). Clearly,

$$\pi(\mathfrak{M}_\gamma)=B(\mathcal{K})\otimes 1_{\overline{\mathcal{K}}}\otimes\mathfrak{M}_{\gamma,\mathrm{l}}^{\mathrm{AW}}.$$

Set $J:=J_{\mathcal{K}}\otimes\Gamma(\epsilon)$, where

$$J_{\mathcal{K}}\Psi_1\otimes\overline{\Psi}_2:=\Psi_2\otimes\overline{\Psi}_1,\qquad\Psi_1,\Psi_2\in\mathcal{K},\tag{128}$$

and ϵ was introduced in (75). Note that

$$J\,B(\mathcal{K})\otimes 1_{\overline{\mathcal{K}}}\otimes\mathfrak{M}_{\gamma,\mathrm{l}}^{\mathrm{AW}}\,J=1_{\mathcal{K}}\otimes B(\overline{\mathcal{K}})\otimes\mathfrak{M}_{\gamma,\mathrm{r}}^{\mathrm{AW}},$$

and if $A\in B(\mathcal{K})\otimes\mathfrak{M}_{\gamma,\mathrm{l}}^{\mathrm{AW}}$, then

$$J\pi(A)J=1_{\mathcal{K}}\otimes\Big(1_{\overline{\mathcal{K}}}\otimes\Gamma(\tau)\;\overline{A}\;1_{\overline{\mathcal{K}}}\otimes\Gamma(\tau)\Big),$$

where τ was introduced in (74).

Proposition 3.

$$\Big(\pi,\ \mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}}),\ J,\ \mathcal{H}_\gamma^+\Big)$$

is a standard representation of $\mathfrak{M}_\gamma$*, where*

$$\mathcal{H}_\gamma^+:=\{\pi(A)J\pi(A)\;B\otimes\Omega\;:B\in B_+^2(\mathcal{K}),A\in\mathfrak{M}_\gamma\}^{\mathrm{cl}}.$$

Set

$$L_{\mathrm{fr}}:=K\otimes 1\otimes 1-1\otimes\overline{K}\otimes 1+1\otimes 1\otimes\mathrm{d}\Gamma(h\oplus(-\overline{h})).$$

Proposition 4. *Assume (125). Then*

$$\begin{aligned}\pi(V_\gamma) &:= 1_{\overline{\mathcal{K}}}\check{\otimes}V_\gamma\\ &= 1_{\overline{\mathcal{K}}}\check{\otimes}a^*\Big((1+\rho)^{\frac12}v,\overline{\rho}^{\frac12}v^\star\Big)+1_{\overline{\mathcal{K}}}\check{\otimes}a\Big((1+\rho)^{\frac12}v,\overline{\rho}^{\frac12}v^\star\Big)\end{aligned}$$

is essentially self-adjoint on finite particle vectors of $\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ *and is affiliated to the* W^**-algebra* $B(\mathcal{K})\otimes 1_{\overline{\mathcal{K}}}\otimes\mathfrak{M}_{\gamma,\rm l}^{\rm AW}$. *Moreover,*

$$\begin{aligned}J\pi(V_\gamma)J &:= 1_{\mathcal{K}}\otimes\Big(1_{\overline{\mathcal{K}}}\otimes\Gamma(\tau)\ \overline{V_\gamma}\ 1_{\overline{\mathcal{K}}}\otimes\Gamma(\tau)\Big)\\ &= 1_{\mathcal{K}}\otimes a^*\Big(\rho^{\frac12}\overline{v}^\star,(1+\overline{\rho})^{\frac12}\overline{v}\Big)+1_{\mathcal{K}}\otimes a\Big(\rho^{\frac12}\overline{v}^\star,(1+\overline{\rho})^{\frac12}\overline{v}\Big).\end{aligned}$$

Set

$$L_\gamma := L_{\rm fr}+\pi(V_\gamma)-J\pi(V_\gamma)J. \tag{129}$$

Theorem 59. 1) $L_{\rm fr}$ *is the standard Liouvillean of the free Pauli-Fierz system* $(\mathfrak{M}_\gamma,\tau_{\rm fr})$.

2) *Suppose that (127) holds. Then* L_γ *is essentially self-adjoint on* $\mathrm{Dom}(L_{\rm fr})\cap\mathrm{Dom}(\pi(V_\gamma))\cap\mathrm{Dom}(J\pi(V_\gamma)J)$ *and is the Liouvillean of the Pauli-Fierz system* $(\mathfrak{M}_\gamma,\tau_\gamma)$.

13.7 Confined Pauli-Fierz systems—standard representation

Again we make the assumption $\mathrm{Tr}\gamma<\infty$ about the confinement of the bosons.

We can use the standard representation $\pi_{\rm l}$ for $B(\mathcal{K})$ in $B(\mathcal{K}\otimes\overline{\mathcal{K}})$ (in the form of Subsection 8.6) and the Araki-Woods representation $\theta_{\gamma,\rm l}$ for $B(\Gamma_{\rm s}(\mathcal{Z}))$ in $B(\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$. Thus we obtain the representation

$$\pi_{\gamma,\rm l}:B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z}))\to B(\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})).$$

defined by

$$\pi_{\gamma,\rm l}(A_1\otimes A_2)=A_1\otimes 1_{\overline{\mathcal{K}}}\otimes\theta_{\gamma,\rm l}(A_2),\quad A_1\in B(\mathcal{K}),\quad A_2\in B(\Gamma_{\rm s}(\mathcal{Z})).$$

Note that

$$\pi_{\gamma,\rm l}(A):=1_{\overline{\mathcal{K}}}\check{\otimes}\pi_\gamma^{\rm semi}(A),\qquad A\in B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})).$$

One can put it in a different way. Introduce the obvious unitary identification

$$\tilde{U}:\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})\otimes\overline{\mathcal{K}}\otimes\Gamma_{\rm s}(\overline{\mathcal{Z}})\to\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}}).$$

Set

$$U_\gamma:=1_{\mathcal{K}\otimes\overline{\mathcal{K}}}\otimes R_\gamma\ \tilde{U}.$$

Then

$$\pi_{\gamma,\rm l}(A)=U_\gamma\ A\otimes 1_{\overline{\mathcal{K}}\otimes\Gamma_{\rm s}(\overline{\mathcal{Z}})}\ U_\gamma^*,\quad A\in B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})).$$

Theorem 60.

$$\pi_{\gamma,\mathrm{l}}\left(B\big(\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z})\big)\right)=\pi(\mathfrak{M}_\gamma),$$

$$\pi_{\gamma,\mathrm{l}}\Big(\mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}\Big)=\pi\left(\tau_\gamma^t(\pi_\gamma^{\mathrm{semi}}(A))\right),\quad A\in B(\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z})),$$

$$L_\gamma=U_\gamma\left(H\otimes 1_{\overline{\mathcal{K}}\otimes\Gamma_{\mathrm{s}}(\overline{\mathcal{Z}})}-1_{\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z})}\otimes\overline{H}\right)U_\gamma^*.$$

Let us stress that in the confined case the Liouvilleans L_γ are unitarily equivalent for different γ.

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